

CHARACTERISTICS OF FAR FIELD AND ENERGY FLOW  
DUE TO A MOVING RADIATING SOURCE IN VARIOUS MEDIA

〔移動輻射源在多種  
介質內發出的遠域場  
及能流的若干特徵〕

---

by

Ng Chung-sang

(吳仲生)

A Thesis Submitted In Partial Fulfillment  
of the Requirements for the Degree of  
Master of Philosophy in Physics

The Chinese University of Hong Kong

May 1988

thesis  
QC  
718.5  
R3 W8

488178



# CHARACTERISTICS OF FAR FIELD AND ENERGY FLOW DUE TO A MOVING RADIATING SOURCE IN VARIOUS MEDIA

<u>CONTENTS</u>	<u>Page</u>
Acknowledgement	iv
Abstract	v
1 Introduction	1
2 Summary of Lai-Chan's method	7
3 Doppler-shifted wave-vector surface (DWS)	12
3.1 Definition and physical meaning	12
3.2 Examples of DWS's	13
3.2.1 vacuum	13
3.2.2 isotropic non-dispersive media	14
3.2.3 non-dispersive uniaxial media	14
3.2.4 isotropic cold plasma	15
3.2.5 cold magnetoplasma	15
3.3 Finding the Gaussian curvature (K) of a DWS having cylindrical symmetry	17
3.4 Finding K as function of $\tau$ for DWS having the form $k_{\rho}^2 = A k_z^2 + B k_z + C$	19
4 The moving radiating source	21
4.1 Definition using relativistic transformation	21
4.2 Examples of moving source of velocity V	22
4.2.1 helically moving charge	22
4.2.2 linearly oscillating charge	23
4.3 Real field for separate terms $\delta(\omega \pm \frac{n\omega_0}{\gamma} - \mathbf{k} \cdot \mathbf{V})$	26
5 Radiation in vacuum	29
5.1 Moving dipole radiation by usual method	29



5.2	Using Lai-Chan's method	31
5.3	Radiation energy	33
6	Radiation in isotropic non-dispersive media	35
6.1	$V < c / \sqrt{\epsilon}$	35
6.2	$V > c / \sqrt{\epsilon}$	37
6.3	Cerenkov radiation from the $\delta(\omega - k \cdot V)$ term	40
7	Radiation in an isotropic cold plasma	43
7.1	The far field and the energy flow	43
7.2	When $\omega_0 < \gamma \omega_p$ : some special features	46
7.2.1	complex Doppler effect	47
7.2.2	all forward radiation	48
7.2.3	anomalous Doppler effect	49
7.2.4	red-shift of forward radiation	49
7.2.5	infinite emitted power at $\theta_c$	49
8	Radiation in non-dispersive uniaxial media	51
8.1	Dispersion relation	51
8.2	$V$ along symmetric axis with $V < c / \sqrt{\epsilon_{\perp}}$	53
8.3	$V$ along symmetric axis with $V > c / \sqrt{\epsilon_{\perp}}$	56
8.4	$V$ not parallel to the symmetric axis	58
9	Cerenkov radiation in an uniaxial cold plasma	64
9.1	Dispersion relation of uniaxial cold plasma	64
9.2	Cerenkov radiation for $V$ along $B$	65
9.3	Compare the radiation power found by J·E method	70
10	Radiation in a cold magnetoplasma	72
10.1	Dispersion relation of a cold magnetoplasma	72
10.2	DWS for $V$ along $B$	78
10.3	Finding the points of stationary phase	81



11 Enhanced field-aligned radiation in a cold magnetoplasma	86
11.1 Condition for $\lambda_\phi = 0$	86
11.2 Formula for finding the radiation field	89
11.3 Group velocity and $\lambda_{kz}$	91
11.4 Integration over $\phi$	93
11.5 Self mode interference of the field-aligned wave	96
12 Conclusions	99
References	101
Figures	102

### Acknowledgements

I wish to express my gratitude to my supervisor, Dr. H.M.Lai, for his patient guidance and many valuable suggestions.

The financial support of the Institute of Science and Technology of the Chinese University of Hong Kong is cordially acknowledged.



## ABSTRACT

Lai-Chan's method is applied to evaluate the far field caused by a uniformly moving source in a number of media. The concept of the Doppler-shifted wave-vector surface (DWS), whose geometric properties (at the point of stationary phase) directly determine the far field, is emphasized. The point of stationary phase on a DWS is found explicitly as a function of space-time variables for some simple media, including a non-dispersive uniaxial medium, an isotropic cold plasma, and an uniaxial cold plasma in which a charge is moving along the field. In the case of isotropic cold plasma, the complex Doppler effect and the all-forward radiation effect is shown with the help of the DWS. For an uniaxial cold magnetoplasma, the DWS for Cerenkov radiation is found to be quite different from that for the non-dispersive case. For a general cold magnetoplasma, with the source moving along B, DWS's are plotted and the points of stationary phase can be found numerically with the help of these graphs ; the far field can be found by straight forward calculation. One special case in the cold magnetoplasma, where the radiation field along B is enhanced to be having  $1/\sqrt{r}$  dependence, is considered. It is found that the existence of this enhanced field-aligned radiation depends on the velocity of the source. The self mode interference effect is also indicated in this case.



---

## CHAPTER I

### INTRODUCTION

It is well-known that the radiation field of a localized monochromatic radiating source in a homogeneous medium can be calculated by carrying out an integration over the wave-vector  $k$ -space after taking the Fourier transform of the Maxwell equation. The main contribution to the far field comes from the wave-vector surface with  $k$  on this surface satisfying the dispersion relation of the medium. Furthermore, through the method of stationary phase (Lighthill, 1960; Giles, 1978), the dominant contribution to the far field at position  $x$  relative to the source has been shown really coming from the surface element around the point of stationary phase on the wave-vector surface where the normal direction of this point is the same as  $x$ . The far field found by this method has the  $1/\sqrt{K}$  - dependence, where  $K$  is the magnitude of the Gaussian curvature at the point of stationary phase.

Lai and Chan has extended the above method to calculate the far field from a localized radiating source moving at velocity  $V$  in an anisotropic dispersive homogeneous and loss-free medium (Lai & Chan, 1986). However, in the case of a moving source of a proper frequency  $\omega_0$ , the frequency is shifted to  $\frac{\omega_0}{\gamma} + k_z V$  by the Doppler effect ; the wave-vector surface, accordingly modified is called the Doppler-shifted wave-vector surface (DWS). Here  $\gamma \equiv (1 - \beta^2)^{-1/2}$  with  $\beta \equiv V/c$  and the  $z$ -axis is taken to be along  $V$ . The aim of this thesis is to study this formulation and to apply it to several kinds



of media to see some of the characteristics of the far field and the energy flow.

In chapter II , the formulation given by Lai and Chan is summarized for the use of later chapters.

The concept of DWS is studied in chapter III. Some examples of DWS's for several media are also given. In Lai and Chan's paper, the angle  $\theta$  between the  $k$  vector and the cylindrically symmetric axis is used as a coordinate of the cylindrically symmetric DWS to calculate the Gaussian curvature of the surface. In this thesis, the variable  $k_z$  (the  $z$ -component of the wave-vector  $k$ ) is used instead of  $\theta$  since it is easier to express the surface as an explicit function of  $k_z$ ,  $\phi$  than in  $\theta$ ,  $\phi$ , where  $\phi$  is the azimuthal angle of the cylindrical coordinate system, especially in the case of a moving source. Also, the derivative  $dk_\rho/dk_z$  has a simple relation to the space-time variables  $(x,t)$  at the observation point, since the normal of the DWS at the point of stationary phase is parallel to the observation direction  $x - Vt$ . Making use of this fact, the points of stationary phase for some simple media, where the corresponding DWS,s are of the quadratic form, can be found as functions of space-time explicitly, and thus the radiation field can be found as explicit function of  $(x,t)$  also. For a DWS having more complicated form, it is found that the graph of DWS is still very useful in the determination of the points of stationary phas.

The definition of a moving radiation source is given in chapter IV through relativistic transformation from the rest frame of the source to the rest frame of the medium. Two simple radiating sources, namely, a helically moving charge



and a linearly oscillating charge, are considered and their Fourier transforms are given in an exact form which are used in the later chapters.

In chapter V, the simplest case of a moving dipole's radiation in vacuum is considered to give a check of the formulation. It is found that the result of Lai-Chan's method is the same as that given by Liénard-Wiechart's method and that found by taking a relativistic transformation of the radiation field due to a stationary dipole.

Isotropic non-dispersive medium is considered in chapter VI. For the velocity of a moving source  $\beta > 1/\sqrt{\epsilon}$ , where  $\epsilon$  is the dielectric constant of the medium, the DWS changes from ellipsoid to hyperboloid and so the far field found in this case is different from that for that in  $\beta < 1/\sqrt{\epsilon}$ . One difference is that there are two points of stationary phase for a given  $(x, t)$ . Also, the far field in this case is found to be divergent on the Cerenkov cone. This unphysical result is due to the fact that nondispersive model is not valid at high frequency. Cerenkov radiation is also considered.

In chapter VII, the far field of a moving source in an isotropic cold plasma is found as a function of space-time variables. The complex Doppler effect (Frank, 1943) exists in this case. Using DWS, this effect is easily seen since the DWS is above the  $k_\rho$  axis if the velocity of the source exceeds a certain value such that  $\omega_0/\gamma < \omega_p$ , where  $\omega_p$  is the plasma frequency of the medium. From the picture of the DWS, the radiation for this case is also easily seen to be all-forward, since now all  $k$  are having positive  $z$ -component  $k_z$ .

A simple anisotropic non-dispersive uniaxial medium is



considered in chapter VIII. For the source moving along the symmetric axis, the DWS for both the ordinary mode and the extraordinary mode are still of the same form as that of the isotropic case, so the radiation field can be found using the similar method (but the detailed form of the radiation field is different of course). For the velocity not along the symmetric axis, the DWS of the extraordinary mode is not cylindrically symmetric any more. However, the Gaussian curvature can still be found by using  $k_z$  and  $\phi$  as coordinate variables of the surface. Since  $\partial k_\rho / \partial k_z$ ,  $\partial k_\rho / \partial \phi$  and  $\phi$  in this case are simply related to  $(x, t)$ , the radiation field in this case can also be found as an explicit function of  $(x, t)$ . Since the DWS in this case is in a general form of a quadratic surface, we can see that the point of stationary phase on a quadratic DWS can be found explicitly as a function of space-time variables.

The dispersion relation of a cold magnetoplasma is discussed in chapter X. Graphs of  $\omega$  vs  $k$  for different  $\theta$  and the DWS for  $V$  parallel to  $B$  are plotted. Since the DWS is complicated in this case, it is difficult to find the point of stationary phase as a function of space-time explicitly. The method to find it numerically by making use of the graph of DWS is discussed and the formulas for the calculation of the group velocity and Gaussian curvature are given.

One special case where the external magnetic field is very strong (i.e. the uniaxial cold magnetoplasma), is considered in chapter IX in connection to the Cerenkov radiation cause by a charge moving along the external magnetic field. In this simple case, the radiation field can be found as a function of  $(x, t)$  and so the radiation power



can be calculated. However, it is found that this power diverges as the velocity of the charge tends to zero. This divergence may be traced to the fact that, as  $\beta \rightarrow 0$ , the radiation comes from waves of infinite  $k$  ; the cold plasma model is expected to fail in this case.

Another special case in a cold magnetoplasma is that, for certain plasma parameters, the cylindrically symmetric DWS is having maximum (or minimum)  $k_z$  for  $k_\rho \neq 0$  . In this case, the principal curvature  $\lambda_\phi$  is equal to zero, so the general formula which depend on the inverse of the Gaussian curvature can not be used. Instead, an integration over  $\phi$  on a ring of these maximum (or minimum) wavevector should be done separately. This results in a far field, which is inversely proportional to the square root of distance, along the external magnetic field. This wave is referred as the enhanced field-aligned wave. The necessary and sufficient condition for the occurrence of this  $\lambda_\phi = 0$  case is found for the  $V = 0$  case. For  $V \neq 0$ , the value for this case to happen can only be found numerically with the help of the graph of corresponding DWS, and this calculation is done for some cases. Examples with using a helically moving charge and a linearly moving charge as the moving radiating source are considered. In the former case, it is found that only the wave with frequency equal to the gyro-frequency of the helically moving charge can survive. All other harmonics vanish after the integration over  $\phi$ . The energy flow due to the "zero frequency" term also vanishes since both radiation  $E$  and  $B$  fields of this term are along  $\hat{z}$  . For the linearly oscillating charge, self-mode interference effect is found except for the charge is oscillating along or perpendicular



to  $\hat{z}$ , so this effect is explained by the interference between the  $k$ 's on the ring of  $\lambda_\phi = 0$ .

Conclusions are given in the last chapter.

## CHAPTER II

### SUMMARY OF LAI - CHAN'S METHOD

It is convenient to write the macroscopic Maxwell equation as (e.g. Landau, 1960, chap.12) :

$$\nabla \cdot \mathbf{B}(\mathbf{x}, t) = 0, \quad \nabla \cdot \mathbf{D}(\mathbf{x}, t) = 4\pi\rho(\mathbf{x}, t) \quad (2-1)$$

$$\nabla \times \mathbf{E}(\mathbf{x}, t) = -\frac{1}{c} \frac{\partial \mathbf{B}(\mathbf{x}, t)}{\partial t} \quad (2-2)$$

$$\nabla \times \mathbf{B}(\mathbf{x}, t) = \frac{1}{c} \frac{\partial \mathbf{D}(\mathbf{x}, t)}{\partial t} + \frac{4\pi}{c} \mathbf{J}(\mathbf{x}, t) \quad (2-3)$$

where  $\mathbf{E}(\mathbf{x}, t)$  and  $\mathbf{B}(\mathbf{x}, t)$  are the macroscopic electric and magnetic field respectively,  $\mathbf{J}(\mathbf{x}, t)$  and  $\rho(\mathbf{x}, t)$  are the external current density and charge density,  $\frac{\partial}{\partial t} \mathbf{D}(\mathbf{x}, t)$  is the displacement current. Taking Fourier-Laplace transform of the above equations and using the boundary condition and initial condition that all field quantities equal to zero either at  $|\mathbf{x}| \rightarrow \infty$  or at the initial time, we get :

$$\mathbf{k} \cdot \mathbf{B}(\mathbf{k}, \omega) = 0, \quad i\mathbf{k} \cdot \bar{\epsilon}(\mathbf{k}, \omega) \mathbf{E}(\mathbf{k}, \omega) = 4\pi\rho(\mathbf{k}, \omega) \quad (2-4)$$

$$\mathbf{k} \times \mathbf{E}(\mathbf{k}, \omega) = \omega \mathbf{B}(\mathbf{k}, \omega) / c \quad (2-5)$$

$$\mathbf{k} \times \mathbf{B}(\mathbf{k}, \omega) = -\frac{\omega}{c} \bar{\epsilon}(\mathbf{k}, \omega) \cdot \mathbf{E}(\mathbf{k}, \omega) - \frac{4\pi}{c} \mathbf{J}(\mathbf{k}, \omega) \quad (2-6)$$

where  $\mathbf{k}, \omega$  are the wave-vector and frequency respectively,  $i$  is  $\sqrt{-1}$  and  $\bar{\epsilon}$  is the dielectric tensor satisfying :

$$\mathbf{D}(\mathbf{k}, \omega) = \bar{\epsilon}(\mathbf{k}, \omega) \cdot \mathbf{E}(\mathbf{k}, \omega) \quad (2-7)$$

by (2-5) and (2-6) , we get :

$$\bar{\Delta}(\mathbf{k}, \omega) \cdot \mathbf{E}(\mathbf{k}, \omega) = -4\pi i \omega \mathbf{J}(\mathbf{k}, \omega) \quad (2-8)$$

where

$$\bar{\Delta}(\mathbf{k}, \omega) \equiv \omega^2 \bar{\epsilon}(\mathbf{k}, \omega) - c^2 k^2 \bar{\mathbf{I}} + c^2 \mathbf{k} \mathbf{k} \quad (2-9)$$



where  $\bar{I}$  is the unit tensor. Taking the inverse transform of (2-8) :

$$E(\mathbf{x}, t) = \int \frac{d\mathbf{k}}{(2\pi)^3} \int_{-\infty+i\epsilon}^{\infty+i\epsilon} \frac{d\omega}{2\pi} \frac{-4\pi i \omega \bar{\Gamma}(\mathbf{k}, \omega)}{\mathcal{D}(\mathbf{k}, \omega)} \cdot J(\mathbf{k}, \omega) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} \quad (2-10)$$

where  $\mathcal{D}(\mathbf{k}, \omega)$  is the determinant of  $\bar{\Delta}(\mathbf{k}, \omega)$ , and  $\bar{\Gamma}$  is the adjoint tensor,

$$\bar{\Gamma}(\mathbf{k}, \omega) \equiv \mathcal{D}(\mathbf{k}, \omega) \bar{\Delta}^{-1}(\mathbf{k}, \omega) , \quad (2-11)$$

$\nu$  is an infinitesimal positive quantity which is introduced to make the result satisfy causality,  $\int d\mathbf{k}$  means volume integration over all  $\mathbf{k}$ -space.

Making use of (2-10), Lai and Chan developed a method to calculation the far field and energy flow radiated from a localized radiating source uniformly moving in an anisotropic dispersive and lossless medium. The Fourier-Laplace transform of the current density of the moving radiating source with velocity  $\mathbf{V}$  is assumed to be monochromatic in the following form (Lai & Chan, 1986):

$$J(\mathbf{k}, \omega) = 2\pi J(\mathbf{k}) \delta\left(\omega - \frac{\omega_0}{\gamma} - \mathbf{k} \cdot \mathbf{V} - i\epsilon\right), \quad (2-12)$$

where  $J(\mathbf{k})$  is a well behaved function of  $\mathbf{k}$  , and  $\gamma$  is the relativistic factor  $\gamma = (1 - V^2/c^2)^{-1/2}$ ,  $\omega_0$  is the frequency of the radiating source in the co-moving frame of the source . Substituting (2-12) into (2-10) :

$$E(\mathbf{x}, t) = \frac{-ie^{-i\omega_0 t/\gamma}}{2\pi^2} \int d\mathbf{k} \left[ \frac{\omega \bar{\Gamma}(\mathbf{k}, \omega)}{\mathcal{D}(\mathbf{k}, \omega)} \right]_{\omega} \cdot J(\mathbf{k}) e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{V}t)} \quad (2-13)$$

where  $[ ]_{\omega}$  denotes that  $\omega = \frac{\omega_0}{\gamma} + \mathbf{k} \cdot \mathbf{V} + i\epsilon$  inside the bracket. For large  $|\mathbf{x} - \mathbf{V}t|$ , the integration is mainly contributed by the wave-vector lying on the surface satisfying



$$\mathcal{D}(k, \frac{\omega_0}{\gamma} + k \cdot V) = 0 \quad (2-14)$$

This surface is called the Doppler-shifted wave-vector surface (DWS). Expanding the  $k$  value around DWS and perform the integration (2-13) for the normal component of the DWS first. A surface integration on the DWS is then left :

$$E(x, t) = \frac{e^{-i\omega_0 t/\gamma}}{\pi} \int_{DWS'} d^2k \left[ \frac{\omega \bar{\Gamma}(k, \omega)}{(\frac{\partial \mathcal{D}}{\partial k} + \frac{\partial \mathcal{D}}{\partial \omega} V)_N} \right] \cdot J(k) e^{i k \cdot (x - \vec{V}t)} \quad (2-15)$$

where DWS' denotes that the surface integration is performed only over that part of DWS satisfying  $(x - \vec{V}t) \cdot \hat{N} > 0$ , with  $\hat{N}$  being the unit vector of the normal direction of the DWS which is defined as the direction  $\partial \omega_0(k)/\partial k$ , and  $( )_N$  denotes the projection of the vector along  $\hat{N}$ . Note that the group velocity of a wave in the medium can be found by :

$$V_E = - \frac{\partial \mathcal{D}}{\partial k} / \frac{\partial \mathcal{D}}{\partial \omega} \quad (2-16)$$

and for the wave-vector on DWS :

$$V_E = \frac{\partial \omega}{\partial k} = \partial \omega_0(k) / \gamma \partial k + V \quad (2-17)$$

So,

$$\frac{\partial \mathcal{D}}{\partial k} + \frac{\partial \mathcal{D}}{\partial \omega} V = - \frac{\partial \mathcal{D}}{\partial \omega} [\partial \omega_0(k) / \gamma \partial k] \quad (2-18)$$

$$= + \frac{\partial \mathcal{D}}{\partial \omega} (V - V_E) \quad (2-19)$$

is along  $\hat{N}$  of the DWS. For large  $|x - \vec{V}t|$ , (2-16) can be further simplified using the method of stationary phase : the main contribution to (2-13) are coming from the wave-vector  $k^0$  on DWS such that  $\hat{N}$  of these points is exactly parallel to  $x - \vec{V}t$  (see Fig.2-1). The final result is :

$$E(x,t) = \sum_{\sigma} \frac{2A^{\sigma}}{|x-Vt|K^{1/2}} \left[ \frac{\omega \bar{\Gamma}(k^{\sigma}, \omega)}{\left( \frac{\partial \mathcal{D}}{\partial k} + \frac{\partial \mathcal{D}}{\partial \omega} V \right)_N} \right]_D \cdot J(k^{\sigma}) e^{i[k^{\sigma} \cdot x - \left( \frac{\omega}{\gamma} + k^{\sigma} \cdot V \right) t]} \quad (2-20)$$

where  $K$  is the magnitude of the Gaussian curvature of the DWS at  $k^{\sigma}$  :

$$K = |\lambda_u \lambda_v| \quad (2-21)$$

with  $\lambda_u$  and  $\lambda_v$  being the principal curvatures;

$$A^{\sigma} = \exp \left[ \frac{i\pi}{4} [\text{sign}(\lambda_u) + \text{sign}(\lambda_v)] \right] \quad (2-22)$$

Note that  $K \neq 0$  is assumed in (2-20). Choosing two coordinates  $u, v$  of the DWS having the following properties :

$$\frac{\partial k}{\partial u} \cdot \frac{\partial k}{\partial v} = 0, \quad \frac{\partial^2 k}{\partial u \partial v} \cdot \hat{N} = 0 \quad (2-23)$$

Then the principal curvatures can be found by :

$$\lambda_u = \left( \frac{\partial^2 k}{\partial u^2} \right)_N / \left| \frac{\partial k}{\partial u} \right|^2 \quad (2-24)$$

$$\lambda_v = \left( \frac{\partial^2 k}{\partial v^2} \right)_N / \left| \frac{\partial k}{\partial v} \right|^2 \quad (2-25)$$

For a single  $\sigma$  term in (2-20), the time average energy flow can also be found by using the well-known expression (e.g. Stix, 1962):

$$\langle S(x,t) \rangle = - \frac{1}{16\pi\omega} E^* \cdot \frac{\partial \bar{\Delta}}{\partial k} \cdot E$$

where  $*$  means complex conjugate. After some calculation :

$$\langle S(x,t) \rangle = \frac{V_E^2 V_E}{4\pi(x-Vt')^2 (V_E - V)^4 K} J^*(k^{\sigma}) \cdot \left[ \frac{\omega \bar{\Gamma}}{\partial \mathcal{D} / \partial \omega} \right]_D \cdot J(k^{\sigma}) \quad (2-26)$$

where  $t'$  is the retarded time (see also Fig.2-1) :

$$V_E(t-t') = x - Vt' \quad (2-27)$$

$$x - Vt = (V_E - V)(t-t') \quad (2-28)$$



So, the power received at  $(x, t)$  per unit solid angle subtended at the source at  $t'$  is :

$$\begin{aligned} \frac{dP(x, t)}{d\Omega} &= (x - vt')^2 |\langle S \rangle| \\ &= \frac{V_E^3}{4\pi (V_E - V)^4 K} J^*(k^\sigma) \cdot \left[ \frac{\omega \bar{\Gamma}}{\partial \mathcal{D} / \partial \omega} \right]_D \cdot J(k^\sigma) \end{aligned} \quad (2-29)$$

This power is not the same as that emitted by the source. The emitted power per unit solid angle at  $t'$  is found to be :

$$\frac{dP'}{d\Omega} = f \frac{dP}{d\Omega} \quad (2-30)$$

where

$$f = 1 - \frac{V}{V_E} \cos \theta - \frac{V}{V_E^2} \sin \theta \frac{dV_E}{d\theta} \quad (2-31)$$

with  $\theta$  being the angle between  $V_E$  and  $V$ .  $f$  has the physical meaning that it is the ratio between the retarded time scale to the observation time scale :

$$f = \frac{\delta t'}{\delta t} \quad (2-32)$$

Note that we have assumed in the above results that  $\frac{\partial \mathcal{D}}{\partial k}$  and  $\frac{\partial \mathcal{D}}{\partial \omega}$  is non-zero. However, for a degenerate medium such that  $\mathcal{D} = \alpha \mathcal{D}'^2$ , e.g. in vacuum  $\mathcal{D} = \omega^2(\omega^2 - c^2 k^2)^2$ , this is not the case. Fortunately, if the adjoint tensor for these degenerate medium having the form of  $\bar{\Gamma} = \alpha \mathcal{D}' \bar{\Gamma}'$ , where  $\bar{\Gamma}'$  is non-zero for  $\mathcal{D} = 0$ , then the above results (2-20), (2-26), (2-29) can still be used if we replace  $\mathcal{D} \rightarrow \mathcal{D}'$  and  $\bar{\Gamma} \rightarrow \bar{\Gamma}'$ , which brings a factor of two larger than the results given by the original equations.

---

## CHAPTER III

### DOPPLER-SHIFTED WAVE-VECTOR SURFACE (DWS)

#### 3.1 Definition and physical meaning

From chapter II, it can be seen that the knowledge of DWS is essential in the calculation of the far field due to a uniformly moving radiating source in a medium. Let  $\omega = \omega(k)$  be the dispersion relation of this medium, then DWS can be defined as the three dimensional surface in the wave-vector space which is formed by the intersection of the four dimensional hyper-surface of  $\omega = \omega(k)$  and the plane of Doppler equation

$$\omega = \frac{\omega_0}{\gamma} + k_{\parallel} V, \quad (3-1-1)$$

where  $k_{\parallel}$  is the component of  $k$  along  $V$ . From (2-14), we know that this surface is in fact given by :

$$\mathcal{D}(k, \frac{\omega_0}{\gamma} + k \cdot V) = 0 \quad (3-1-2)$$

In the case of an isotropic medium,  $\omega$  only depends on the magnitude of  $k$  so that we can draw the dispersion surface on a two dimensional graph as a curve  $\omega = \omega(k)$ . The DWS in this case can also be drawn as a curve  $k_{\parallel}$  vs  $k_{\perp}$  which actually is a cylindrical symmetric surface about the direction of  $V$  (see Fig.3-1).

For an anisotropic medium, DWS can not be drawn as a curve in two dimensional space as before in general, except that the medium is cylindrically symmetric and  $V$  is along the axis of symmetry. In this case,  $\omega = \omega(k)$  can only be drawn as a surface in three dimensional space  $\omega = \omega(k_{\parallel}, k_{\perp})$  like that of Fig. 3-2 but the DWS can still be drawn as a two



dimensional curve like that of Fig. 3-1b.

The physical meaning of DWS is that for a radiating source of single proper frequency and uniformly moving through a medium, the frequency of the radiated wave is no longer single frequency, but depend on the direction of the wave-vector  $k$ . Note that only the wave with frequency and wave-vector lying on the dispersion surface  $\omega = \omega(k)$  can propagate through the medium. The frequency of the radiated wave of the wave-vector on DWS can be found easily by (3-1-1). The direction of the group velocity of the radiated wave and the retarded position of the radiating source can also be found by the method in Chapter I with the help of the graph of DWS (see Fig. 2-1). Also, the  $K^{-1}$  dependence of the energy (2-18) indicates the concentration of the wave-vector radiated in a given direction.

Note that if  $V = 0$ , DWS reduces to the ordinary wave-vector surface for a single frequency source.

### 3.2 Examples of DWS

Some simple examples of DWS are given here to illustrate the concept of DWS and some of them will be useful in later discussion.

#### 3.2.1 vacuum

The dispersion relation of vacuum is :

$$\omega^2 = c^2 k^2 \quad (3-2-1)$$

Thus, the DWS is:

$$\begin{aligned} \left( \frac{\omega_0}{\gamma} + c\beta k_z \right)^2 &= c^2 k^2 = c^2 k_\rho^2 + c^2 k_z^2 \\ c^2 k_\rho^2 &= \left( \frac{\omega_0}{\gamma} \right)^2 + 2 \frac{\omega_0}{\gamma} \beta c k_z - \left( \frac{c k_z}{\gamma} \right)^2 \end{aligned} \quad (3-2-2)$$

which is an ellipsoid. See Figs. 3-3, 3-4.  $\beta$  is defined to be

$V/c$  as usual.

The DWS of vacuum can also be written in the form of  $k$  vs  $\theta$  ( angle between  $k$  and  $V$  ) :

$$\omega = ck = \omega_0 / \gamma(1 - \beta \cos \theta) \quad (3-2-3)$$

### 3.2.2 isotropic non-dispersive media

Let the dispersion surface be :

$$\varepsilon \omega^2 = c^2 k^2 \quad (3-3-4)$$

where  $\varepsilon$  is the dielectric constant of the medium usually larger than 1 . So, the DWS can be found in a similar way :

$$c^2 k_\rho^2 = \varepsilon \left( \frac{\omega}{\gamma} \right)^2 + 2 \frac{\omega}{\gamma} \varepsilon \beta c k_z - (1 - \varepsilon \beta^2) c^2 k_z^2 \quad (3-3-5)$$

$$ck = \sqrt{\varepsilon} \omega_0 / \gamma (1 - \beta \sqrt{\varepsilon} \cos \theta) \quad (3-3-6)$$

For  $\beta < 1/\sqrt{\varepsilon}$  , it is an ellipsoid similar to that of vacuum.

For  $\beta = 1/\sqrt{\varepsilon}$  , it becomes a paraboloid.

For  $\beta > 1/\sqrt{\varepsilon}$  , it is a hyperboloid. See Fig. 3-3.

So, it is clear that if the velocity of the radiating source exceeds the velocity of light of that medium, the topology of the DWS is much different from that of vacuum. As we shall see, certain special effects, including Cerenkov radiation appears in such situation.

### 3.2.3 non-dispersive uniaxial media

The dispersion tensor for a wave in a non-dispersive uniaxial medium expressed in cylindrical coordinate  $\hat{\rho}, \hat{\phi}, \hat{z}$  is:

$$\bar{\Delta}(k, \omega) = \begin{matrix} & \hat{\rho} & \hat{\phi} & \hat{z} \\ \begin{matrix} \hat{\rho} \\ \hat{\phi} \\ \hat{z} \end{matrix} & \begin{bmatrix} \omega^2 \varepsilon_\perp - c^2 k_z^2 & 0 & c^2 k_\rho k_z \\ 0 & \omega^2 \varepsilon_\perp - c^2 k^2 & 0 \\ c^2 k_\rho k_z & 0 & \omega^2 \varepsilon_\parallel - c^2 k_\rho^2 \end{bmatrix} \end{matrix} \quad (3-2-7)$$

where  $\hat{z}$  has been chosen as the direction of the symmetric axis of the medium. The dispersion relation is :

$$\mathcal{D} = \omega^2 (\omega^2 \varepsilon_\perp - c^2 k^2) (\omega^2 \varepsilon_\parallel - \varepsilon_\parallel c^2 k_z^2 - \varepsilon_\perp c^2 k_\rho^2) = 0 \quad (3-2-8)$$



The ordinary mode  $\omega^2 \epsilon_{\perp} = c^2 k^2$  is just like (3-2-4), so the DWS of it is similar to (3-2-5), (3-2-6) :

$$c^2 k_{\rho}^2 = \epsilon_{\perp} \left( \frac{\omega}{\gamma} \right)^2 + 2 \frac{\omega}{\gamma} \epsilon_{\perp} \beta c k_z - (1 - \epsilon_{\perp} \beta^2) c^2 k_z^2 \quad (3-2-9)$$

For the extraordinary mode  $\omega^2 \epsilon_{\parallel} \epsilon_{\perp} - \epsilon_{\parallel} c^2 k_z^2 - \epsilon_{\perp} c^2 k_{\rho}^2 = 0$ , the DWS is also of a similar form if the velocity of the source  $V$  is along  $\hat{z}$  :

$$c^2 k_{\rho}^2 = \epsilon_{\parallel} \left( \frac{\omega}{\gamma} \right)^2 + 2 \frac{\omega}{\gamma} \epsilon_{\perp} \beta c k_z - \frac{\epsilon_{\parallel}}{\epsilon_{\perp}} (1 - \epsilon_{\perp} \beta^2) c^2 k_z^2 \quad (3-2-10)$$

So, we see that if  $\beta > \sqrt{\epsilon_{\perp}}$ , DWS of both modes will change from ellipsoid to hyperboloid.

#### 3.2.4 isotropic cold plasma

The dispersion relation is :

$$\omega^2 = \omega_p^2 + c^2 k^2 \quad (3-2-11)$$

where  $\omega_p$  is the plasma frequency. The DWS is given by :

$$c^2 k_{\rho}^2 = \left( \frac{\omega}{\gamma} \right)^2 - \omega_p^2 + 2 \frac{\omega}{\gamma} \beta c k_z - \left( \frac{c k_z}{\gamma} \right)^2 \quad (3-2-12)$$

which is also an ellipsoid. See Fig.3-5.

Note that when velocity  $\beta > n_o$ , where

$$n_o \equiv \sqrt{1 - (\omega_p / \omega_o)^2} \quad (3-2-13)$$

$$\text{i.e. } \frac{\omega}{\gamma} < \omega_p, \quad (3-2-14)$$

the whole DWS is above the  $k_z = 0$  axis. This brings some interesting effect to the radiation field which will be discuss at chapter VII.

#### 3.2.5 cold magneto-plasma

The dispersion tensor of magneto cold plasma (e.g. Stix, 1962) in the  $\hat{\rho}$ ,  $\hat{\phi}$ ,  $\hat{z}$  is :

$$\bar{\Delta} = \begin{pmatrix} \omega^2 S - c^2 k_z^2 & -i\omega^2 D & c^2 k_{\rho} k_z \\ i\omega^2 D & \omega^2 S - c^2 k^2 & 0 \\ c^2 k_{\rho} k_z & 0 & \omega^2 P - c^2 k_{\rho}^2 \end{pmatrix} \quad (3-2-15)$$

where  $\hat{z}$  axis is chosen as the direction of the magnetic field  $B = B \hat{z}$ , and

$$S = 1 - \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2} - \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2}, \quad (3-2-16)$$

$$P = 1 - \left[ \frac{\omega_{pe}^2 + \omega_{pi}^2}{\omega^2} \right] \quad (3-2-17)$$

$$D = - \frac{\omega_{pe}^2 \Omega_e}{\omega(\omega^2 - \Omega_e^2)} + \frac{\omega_{pi}^2 \Omega_i}{\omega(\omega^2 - \Omega_i^2)} \quad (3-2-18)$$

$$R = S + D, \quad L = S - D \quad (3-2-19)$$

with  $\omega_{pe}$  and  $\omega_{pi}$  being plasma frequencies for elections and ions :

$$\omega_{pe}^2 = \frac{4\pi n_e e^2}{m_e}; \quad \omega_{pi}^2 = \frac{4\pi n_i (Ze)^2}{m_i} \quad (3-2-20)$$

and  $\Omega_e$  and  $\Omega_i$  are gyro-frequencies of elections and ions :

$$\Omega_e = \frac{|e|B}{m_e c}; \quad \Omega_i = \frac{|Ze|B}{m_i c} \quad (3-2-21)$$

with  $n_e$ ,  $n_i$  being the number density ;  $m_e$ ,  $m_i$  being the mass of the particles ; and  $Ze$  is the charge of one ion.

The dispersion function is found to be :

$$\begin{aligned} \mathcal{D}(k, \omega) &= \det \bar{\Delta} \\ &= \omega^6 PRL - \omega^4 (SP + RL) c^2 k_\rho^2 - 2\omega^4 SP c^2 k_z^2 + \omega^2 S c^4 k_\rho^4 \\ &\quad + \omega^2 P c^4 k_z^4 + \omega^2 (S + P) c^4 k_\rho^2 k_z^2 \end{aligned} \quad (3-2-22)$$

The dispersion relation  $\mathcal{D}(k, \omega) = 0$  can also be rewritten as the following various forms :

$$\begin{aligned} c^2 k^2 &= \\ &\omega^2 \left[ RL \sin^2 \theta + SP(1 + \cos^2 \theta) \pm \sqrt{(RL - SP)^2 \sin^4 \theta + 4P^2 D^2 \cos^2 \theta} \right] \\ &\quad / 2(S \sin^2 \theta + P \cos^2 \theta) \end{aligned} \quad (3-2-23)$$

$$c^2 k_z^2 = \omega^2 S - \frac{1}{2} \left( 1 + \frac{S}{P} \right) c^2 k_\rho^2 \pm$$



$$\sqrt{\frac{1}{4}\left(1-\frac{S}{P}\right)^2 c^4 k_\rho^4 + \omega^2 D^2 (\omega^2 - c^2 k_\rho^2 / P)} \quad (3-2-24)$$

$$c^2 k_\rho^2 = \frac{1}{2} \left\{ \omega^2 \left( P + S - \frac{D^2}{S} \right) - \left( 1 + \frac{P}{S} \right) c^2 k_z^2 \pm \right.$$

$$\left. \sqrt{\left[ \omega^2 \left( P + S - \frac{D^2}{S} \right) - \left( 1 + \frac{P}{S} \right) c^2 k_z^2 \right]^2 - \frac{4P}{S} \left[ \omega^4 (S^2 - D^2) - (2\omega^2 S - c^2 k_z^2) c^2 k_z^2 \right]} \right\} \quad (3-2-25)$$

Note that the wavevector surface is cylindrically symmetric about  $\hat{z}$ , i.e. the radial component of the wave-vector  $k_\rho$  can be expressed as function of  $k_z$  and  $\omega$  :

$$k_\rho^2 = f(\omega^2, k_z^2) . \quad (3-2-26)$$

Then the DWS can be easily plotted on a two dimensional graph as  $k_\rho$  vs  $k_z$  if the velocity of the radiation source is along  $\hat{z}$  :

$$k_\rho^2 = f\left[\left(\frac{\omega_0}{\gamma} + ck_z \beta\right)^2, k_z^2\right] . \quad (3-2-27)$$

See Fig.3-6.

### 3.3 Finding the Gaussian curvature (K) of the a DWS having cylindrical symmetry

If the DWS has cylindrically symmetric and  $k_\rho$  can be expressed as a function of  $k_z$ , i.e.

$$\mathbf{k} = k_\rho(k_z) \left[ \cos\phi \hat{x} + \sin\phi \hat{y} \right] + k_z \hat{z} , \quad (3-3-1)$$

then we can use  $\phi$  and  $k_z$  as coordinate variables to calculate the Gaussian curvature by (2-21). We now show that

(2-23) is satisfied with respect to  $\phi$  and  $k_z$ . First,

$$\frac{\partial \mathbf{k}}{\partial k_z} = k'_\rho (\cos\phi \hat{x} + \sin\phi \hat{y}) + \hat{z} , \quad (3-3-2)$$

$$\frac{\partial \mathbf{k}}{\partial \phi} = k_\rho (-\sin\phi \hat{x} + \cos\phi \hat{y}) , \quad (3-3-3)$$

$$\text{where } k'_\rho \equiv dk_\rho / dk_z . \quad (3-3-4)$$

This immediately leads to

$$\frac{\partial k}{\partial k_z} \cdot \frac{\partial k}{\partial \phi} = 0 \quad (3-3-5)$$

Secondly, the unit vector  $\hat{N}$  which is of the same direction of  $V_E - V$  (or  $x - Vt$ ) can be calculated by :

$$\begin{aligned} \hat{N} &= \pm \frac{\frac{\partial k}{\partial \phi} \times \frac{\partial k}{\partial k_z}}{\left| \frac{\partial k}{\partial \phi} \times \frac{\partial k}{\partial k_z} \right|} \\ &= \pm \frac{\cos \phi \hat{x} + \sin \phi \hat{y} - k'_\rho \hat{z}}{\sqrt{1 + k'^2_\rho}} \end{aligned} \quad (3-3-6)$$

where the sign + or - should be determined by the direction of  $V_E - V$ . So, the normal component of the mixed double derivative is equal to zero :

$$\left( \frac{\partial^2 k}{\partial k_z \partial \phi} \right)_N = k'_\rho (-\sin \phi \hat{x} + \cos \phi \hat{y}) \cdot \hat{N} = 0 \quad (3-3-7)$$

Thus we can apply (2-25) to find the principal curvatures :

$$\lambda_{k_z} = \frac{(\partial^2 k / \partial k_z^2) \cdot \hat{N}}{\left| \partial k / \partial k_z \right|^2} = \pm k''_\rho / (1 + k'^2_\rho)^{3/2} \quad (3-3-8)$$

$$\lambda_\phi = \frac{(\partial^2 k / \partial \phi^2) \cdot \hat{N}}{\left| \partial k / \partial \phi \right|^2} = \mp 1/k_\rho (1 + k'^2_\rho)^{1/2}, \quad (3-3-9)$$

The magnitude of the Gaussian curvature can be obtained :

$$K = |\lambda_{k_z} \lambda_\phi| = |k''_\rho| / k_\rho (1 + k'^2_\rho)^2 \quad (3-3-10)$$

$$\text{where } k''_\rho \equiv \partial^2 k / \partial k_z^2 \quad (3-3-11)$$

Note that  $k'_\rho$  at the stationary phase point has a direct relation with the space-time variables  $(x, t)$  : let

$$x = \rho \hat{\rho} + z \hat{z} \quad (3-3-12)$$

Then by the fact that  $x - Vt \parallel \hat{N}$  and (3-3-6) gives :

$$k'_\rho = \pm (Vt - z) / \rho \quad (3-3-13)$$

where the choice of the  $\pm$  sign should be determined by the consideration of the specific problem. We see that  $k'_\rho$  depends



linearly on the observation time for a fixed observation position; and for any fixed time, it gives the angular dependence which is meaningful even for  $V = 0$ . It turns out that it is a very important parameter, so we define :

$$\tau \equiv (Vt - z) / \rho \quad (3-3-14)$$

As we shall see shortly, the variable  $k^\sigma$  and Gaussian curvature  $K$  at the point of stationary phase can be expressed in terms of  $\tau$ , and thus the far field (2-20) can be found as an explicit function of space and time analytically in certain important cases.

### 3.4 Finding $K$ as function of $\tau$ for DWS having

$$\text{the form } k_\rho^2 = A k_z^2 + B k_z + C$$

One kind of DWS that its quantities at the point of stationary phase, such as  $k$  and  $K$ , can be found explicitly as a function of  $\tau$  is of this form :

$$k_\rho^2 = A k_z^2 + B k_z + C \quad (3-4-1)$$

So,

$$\tau = \pm \frac{dk_\rho}{dk_z} = \pm (2Ak_z + B) / 2k_\rho \quad (3-4-2)$$

Substituting (3-4-1) into (3-4-2) and taking square, we have

$$4\tau^2(C + Bk_z + Ak_z^2) = B^2 + 4ABk_z + 4A^2k_z^2 \quad (3-4-3)$$

The solution of  $k_z$  is :

$$k_z = \frac{1}{2A} \left[ -B \pm \tau \sqrt{\frac{\Delta}{\tau^2 - A}} \right] \quad (3-4-4)$$

where

$$\Delta \equiv B^2 - 4AC \quad (3-4-5)$$

using (3-4-2) again :

$$k_\rho = \frac{1}{2} \sqrt{\frac{\Delta}{\tau^2 - A}} \quad (3-4-6)$$

Note that the  $\phi$  coordinate of the point of stationary phase may have a  $\pi$ -shift from that of  $x - Vt$  if the - sign of (3-4-2) is used.

$\omega$  and  $k$  can also be found :

$$\omega = \frac{\omega_0}{\gamma} + c\beta k_z = \frac{\omega_0}{\gamma} + \frac{c\beta}{2A} \left[ -B \pm \tau \sqrt{\frac{\Delta}{\tau^2 - A}} \right] \quad (3-4-7)$$

$$k = \sqrt{k_\rho^2 + k_z^2} \quad (3-4-8)$$

By (3-3-11) and (3-3-12) :

$$k_\rho'' = (A - \tau^2) / k_\rho \quad (3-4-9)$$

So, the curvatures can be found by (3-3-8) - (3-3-10) :

$$\lambda_{k_z} = \pm k_\rho'' / (1 + k_\rho'^2)^{3/2} = \pm \frac{A - \tau^2}{k_\rho (1 + \tau^2)^{3/2}} \quad (3-4-10)$$

$$\lambda_\phi = \mp 1/k_\rho (1 + \tau^2)^{1/2} \quad (3-4-11)$$

$$K = |\lambda_{k_z} \lambda_\phi| = \frac{|\tau^2 - A|}{k_\rho^2 (1 + \tau^2)^2} \quad (3-4-12)$$

Note that we have assume that  $A \neq 0$  in the above equations.

However, for  $A = 0$ , similar results can be obtained :

$$k_\rho^2 = B k_z + C \quad (3-4-13)$$

$$\tau = \pm B / 2k_\rho \quad (3-4-14)$$

$$k_\rho = \pm B / 2\tau \quad (3-4-15)$$

$$k_z = \frac{B}{4\tau^2} - \frac{C}{B} \quad (3-4-16)$$

$$k_\rho'' = -B^2 / 4k_\rho^3 = \mp 2\tau^3 / B \quad (3-4-17)$$

$$\lambda_{k_z} = \mp 2\tau^3 / B(1 + \tau^2)^{3/2} \quad (3-4-18)$$

$$\lambda_\phi = -2\tau / B(1 + \tau^2)^{1/2} \quad (3-4-19)$$

$$K = [2\tau^2 / B(1 + \tau^2)]^2 \quad (3-4-20)$$



## CHAPTER IV

### THE MOVING RADIATING SOURCE

#### 4.1 Definition using relativistic transformation

Up to now, we only consider the current density of a moving radiation source having the form (2-12). For a more general case of an arbitrary localized radiating source, we can make a Fourier transformation in time and consider each Fourier component separately. So, the charge density and the current density for each component can be written as follows in a certain inertial frame  $S'$  :

$$\rho'(x', t') = \rho(x') e^{-i\omega_0 t'} \quad (4-1-1)$$

$$J'(x', t') = J(x') e^{-i\omega_0 t'} \quad (4-1-2)$$

Then, the charge density and current density for this source moving in a constant velocity can be found by Lorentz transformation from  $S'$  to an inertial frame  $S$  moving in  $-V$  with respect to  $S'$  :

$$\rho(x, t) = \gamma(\rho' + \beta J'_z / c) \quad (4-1-3)$$

$$J_z(x, t) = \gamma(J'_z + V\rho') \quad (4-1-4)$$

$$J_\perp(x, t) = J'_\perp \quad (4-1-5)$$

where  $V = V \hat{z}$  has been assumed. Then,

$$t' = \gamma(t - \beta z/c), \quad z' = \gamma(z - Vt) \quad (4-1-6)$$

$$J(x', t') = \{ J_\perp[x, y, \gamma(z - Vt)] + \gamma(J'_z + \rho' V) \} e^{-i\omega_0 \gamma(t - \frac{\beta z}{c})} \quad (4-1-7)$$

$$= \{ J_\perp(x - Vt) + \gamma[J'_z(x - Vt) + \rho'(x - Vt)V] \hat{z} \} \cdot \exp \left[ i\omega_0 \gamma \frac{\beta(z - Vt)}{c} - i\frac{\omega_0}{\gamma} t \right] \quad (4-1-8)$$

$$\equiv J(x - Vt) e^{-i\omega_0 t/\gamma} \quad (4-1-9)$$

The Fourier transform of (4-1-9) is :

$$J(k, \omega) = \int dx dt J(x - Vt) e^{-i\omega_0 t/\gamma} e^{-i(k \cdot x - \omega t)}$$

$$\begin{aligned}
&= \int dt \exp i \left( \omega - \frac{\omega_0}{\gamma} - \mathbf{k} \cdot \mathbf{V} \right) t \int d(\mathbf{x} - \mathbf{V}t) J(\mathbf{x} - \mathbf{V}t) e^{-i\mathbf{k} \cdot (\mathbf{x} - \mathbf{V}t)} \\
&= 2\pi J(\mathbf{k}) \delta \left( \omega - \frac{\omega_0}{\gamma} - \mathbf{k} \cdot \mathbf{V} \right)
\end{aligned} \tag{4-1-10}$$

Equation (4-1-10) is just (2-12), where

$$J(\mathbf{k}) \equiv \int d\mathbf{x} J(\mathbf{x}) e^{-i\mathbf{k} \cdot \mathbf{x}} \tag{4-1-11}$$

Note that  $J(\mathbf{x})$  is the current density at  $t = 0$ . In conclusion, an arbitrary localized radiation source moving at constant velocity can be expressed as a summation of terms of the form (2-12) and its current density can be found by using relativistic transformation.

## 4.2 Examples of moving radiating sources with velocity $\mathbf{V}$

Two simple one-particle sources will be considered in this chapter to illustrate the method and they will be applied to the calculation in later chapters.

### 4.2.1 helically moving charge

In the "rest" frame  $S'$  of a helically moving charge  $q$ , the charge is in circular motion in the  $x$ - $y$  plane. The charge density and current density of this source is therefore :

$$\rho'(\mathbf{x}', t') = q \delta[\mathbf{x}' - \mathbf{r}'(t')] \tag{4-2-1}$$

$$\mathbf{J}'(\mathbf{x}', t') = q \dot{\mathbf{r}}'(t') \delta[\mathbf{x}' - \mathbf{r}'(t')] \tag{4-2-2}$$

where

$$\mathbf{r}'(t') = \frac{v_{\perp}}{\Omega} (\sin \Omega t' \hat{x} + \cos \Omega t' \hat{y}) \tag{4-2-3}$$

$$\dot{\mathbf{r}}'(t') = \frac{d\mathbf{r}'}{dt'} = v_{\perp} (\cos \Omega t' \hat{x} - \sin \Omega t' \hat{y}) \tag{4-2-4}$$

This situation can be found in the case of a charge confined in a constant magnetic field  $\mathbf{B} = B \hat{z}$ , so that,

$$\Omega = qcB/\mathcal{E} \tag{4-2-5}$$

with  $\mathcal{E}$  being the energy (including the rest energy) of the charge.



Since  $J'_z = 0$ , we have in the laboratory frame S,

$$\begin{aligned}\rho(x,t) &= \gamma\rho = \gamma q\delta(x' - \frac{v_{\perp}}{\Omega}\sin\Omega t')\delta(y' - \frac{v_{\perp}}{\Omega}\cos\Omega t')\delta(z') \\ &= q\delta[x-r(t)]\end{aligned}\quad (4-2-6)$$

where

$$r(t) = \frac{v_{\perp}}{\Omega}(\sin\frac{\Omega}{\gamma}t \hat{x} + \cos\frac{\Omega}{\gamma}t \hat{y}) + Vt \hat{z} \quad (4-2-7)$$

and

$$J_z(x,t) = \gamma\rho'V = qV\delta[x-r(t)] \quad (4-2-8)$$

$$\begin{aligned}J(x,t) &= q[\frac{v_{\perp}}{\gamma}(\cos\frac{\Omega}{\gamma}t \hat{x} - \sin\frac{\Omega}{\gamma}t \hat{y}) + V\hat{z}] \delta[x-r(t)] \\ &= q\dot{r}(t)\delta[x-r(t)]\end{aligned}\quad (4-2-9)$$

The Fourier transform is :

$$\begin{aligned}J(k,\omega) &= \int dx dt q\dot{r}\delta[x-r(t)]e^{-i(k\cdot x - \omega t)} = \\ &= q\int_{-\infty}^{\infty} dt [\frac{v_{\perp}}{\gamma}(\cos\frac{\Omega}{\gamma}t \hat{x} - \sin\frac{\Omega}{\gamma}t \hat{y}) + V\hat{z}] e^{i[\omega t - k\cdot r(t)]}\end{aligned}\quad (4-2-10)$$

Let

$$k = k_{\rho}(\cos\phi \hat{x} + \sin\phi \hat{y}) + k_z \hat{z} \quad (4-2-11)$$

the phase factor  $\exp[-ik\cdot r]$  can be expressed in terms of Bessel functions  $J_n$  :

$$\begin{aligned}e^{-ik\cdot r} &= \exp[-(ik_{\rho}v_{\perp}/\Omega)\sin(\frac{\Omega}{\gamma}t + \phi)] \\ &= \sum_{n=-\infty}^{\infty} J_n(k_{\rho}v_{\perp}/\Omega) e^{-in(\frac{\Omega}{\gamma}t + \phi)}\end{aligned}\quad (4-2-12)$$

Thus,

$$J(k,\omega) = \sum_{n=-\infty}^{\infty} 2\pi J_n(k) \delta(\omega - \frac{n\Omega}{\gamma} - k\cdot V) \quad (4-2-13)$$

where

$$\begin{aligned}J_n(k) &\equiv q[\frac{v_{\perp}}{2\gamma}[e^{-i\phi}J_n(k_{\rho}v_{\perp}/\Omega)(\hat{x}+i\hat{y}) + e^{i\phi}J_{n-1}(k_{\rho}v_{\perp}/\Omega)(\hat{x}-i\hat{y})] \\ &\quad + VJ_n(k_{\rho}v_{\perp}/\Omega)\hat{z}]e^{-in\phi}\end{aligned}\quad (4-2-14)$$

should not be confused with the Bessel function  $J_n$ . Note (4-2-13) is a linear combination of the form (2-12).

#### 4.2.2 linearly oscillating charge

In the "rest" frame,  $\rho'$  and  $J$  are still given by (4-2-1) and (4-2-2) with

$$r(t') = r'_0 \sin \omega_0 t' \quad (4-2-15)$$

where  $r'_0 = (x_0, y_0, z_0)$  is a constant vector.

$$\dot{r}'_0(t') = \omega_0 r'_0 \cos \omega_0 t' \quad (4-2-16)$$

The charge density in S frame is given by (4-1-3).

$$\begin{aligned} \rho(x, t) = & \gamma q \left[ 1 + \frac{\beta \omega_0 z_0}{c} \cos \omega_0 \gamma \left( t - \frac{\beta z}{c} \right) \right] \delta \left[ x - x_0 \sin \omega_0 \gamma \left( t - \frac{\beta z}{c} \right) \right] \\ & \cdot \delta \left[ y - y_0 \sin \omega_0 \gamma \left( t - \frac{\beta z}{c} \right) \right] \delta \left[ \gamma (z - Vt) - z_0 \sin \omega_0 \gamma \left( t - \frac{\beta z}{c} \right) \right] \end{aligned} \quad (4-2-17)$$

In order to find the trajectory of the particle, the third  $\delta$ -function in (4-2-17) must be solved by :

$$\delta[f(z, t)] = \delta[z - z(t)] / \left| \frac{\partial f}{\partial z} \right|_{z=z(t)} \quad (4-2-18)$$

or

$$z(t) = Vt + \frac{z_0}{\gamma} \sin \omega_0 \gamma \left[ t - \frac{\beta z(t)}{c} \right] \quad (4-2-19)$$

where

$$\begin{aligned} f(z, t) & \equiv \gamma (z - Vt) - z_0 \sin \omega_0 \gamma \left( t - \frac{\beta z}{c} \right) , \\ \frac{\partial f}{\partial z} & = \gamma \left[ 1 + \frac{\beta \omega_0 z_0}{c} \cos \omega_0 \gamma \left( t - \frac{\beta z}{c} \right) \right] \end{aligned} \quad (4-2-20)$$

$z(t)$  can be solved by this equation in principle, although an explicit function expressed in  $t$  can not be found. Note that we can rewrite (4-2-17) as :

$$\rho(x, t) = q \delta[x - r(t)] \quad (4-2-21)$$

where

$$r(t) = r_0 \sin \left[ \omega_0 \gamma \left( t - \frac{\beta z(t)}{c} \right) \right] + Vt \hat{z} \quad (4-2-22)$$

$$r_0 \equiv (x_0, y_0, z_0 / \gamma) \quad (4-2-23)$$

$J(x, t)$  can be found in a similar way :

$$J(x, t) = q \dot{r}(t) \delta[x - r(t)] \quad (4-2-24)$$

If  $\omega_0 z_0 \ll c$  then by (4-2-19),

$$z(t) \cong Vt + \frac{z_0}{\gamma} \sin \frac{\omega_0}{\gamma} t \quad (4-2-25)$$



$$\sin \omega_0 \gamma [t - \frac{\beta z(t)}{c}] \cong \sin \frac{\omega_0}{\gamma} t \quad (4-2-26)$$

So,

$$r(t) \cong r_0 \sin \frac{\omega_0}{\gamma} t + Vt \hat{z} \quad (4-2-27)$$

$$J(x, t) \cong q [\frac{\omega_0}{\gamma} r_0 \cos \frac{\omega_0}{\gamma} t + V \hat{z}] \delta [x - r_0 \sin \frac{\omega_0}{\gamma} t - Vt \hat{z}] \quad (4-2-28)$$

$$J(k, \omega) = q \int dt [\frac{\omega_0}{\gamma} r_0 \cos \frac{\omega_0}{\gamma} t + V \hat{z}] \cdot \exp -i[k \cdot r_0 \sin \frac{\omega_0}{\gamma} t + k \cdot Vt - \omega t] \quad (4-2-29)$$

using

$$\exp(-ik \cdot r_0 \sin \frac{\omega_0}{\gamma} t) = \sum_{n=-\infty}^{\infty} J_n(k \cdot r_0) e^{-in\omega_0 t/\gamma} \quad (4-2-30)$$

$$J(k, \omega) = \sum_{n=-\infty}^{\infty} 2\pi J_n(k) \delta(\omega - n\frac{\omega_0}{\gamma} - k \cdot V) \quad (4-2-31)$$

where

$$\begin{aligned} J_n(k) &= \frac{q\omega_0}{2\gamma} r_0 [J_{n+1}(k \cdot r_0) + J_{n-1}(k \cdot r_0)] + qV J_n(k \cdot r_0) \\ &= q \left[ \frac{n\omega_0 r_0}{\gamma k \cdot r_0} + V \right] J_n(k \cdot r_0) \end{aligned} \quad (4-2-32)$$

where the following formula of Bessel function has been used:

$$J_{n-1}(z) + J_{n+1}(z) = 2nJ_n(z)/z \quad (4-2-33)$$

Actually, the exact  $J(k, \omega)$  in this case can be found by relativistic transformation if we note that  $J'(k', \omega')$  and  $\rho'(k', \omega')$  in S frame can be found in a similar way as above :

$$\rho(k', \omega') = 2\pi q \sum_n J_n(k' \cdot r'_0) \delta(\omega' - n\omega_0) \quad (4-2-34)$$

$$J(k', \omega') = 2\pi q \sum_n \frac{n\omega_0 r'_0}{k' \cdot r'_0} J_n(k' \cdot r'_0) \delta(\omega' - n\omega_0) \quad (4-2-35)$$

by the relativistic transformation of  $k'$ ,  $\omega'$ :

$$\omega' = \gamma(\omega - \beta c k_z) \quad (4-2-36)$$

$$k'_z = \gamma(k_z - \beta \omega/c) \quad (4-2-37)$$

$$k'_\perp = k_\perp \quad (4-2-38)$$

So,

$$k' \cdot r'_0 = k \cdot r_0 + \gamma \beta (k_z \beta - \omega/c) z_0$$

$$= \mathbf{k} \cdot \mathbf{r}_0 - (n\beta\omega_0 z_0/c) \quad (4-2-39)$$

where the second equality is due to the presence of

$$\delta(\omega' - n\omega_0) = \frac{1}{\gamma} \delta(\omega - \frac{n\omega_0}{\gamma} - c\beta k_z) \quad (4-2-40)$$

Finally,

$$\begin{aligned} J_z(\mathbf{k}, \omega) &= \gamma [J_z(\mathbf{k}', \omega') + c\beta \rho'(\mathbf{k}', \omega')] \\ &= 2\pi q \sum_{n=-\infty}^{\infty} \left[ \frac{c\beta \mathbf{k} \cdot \mathbf{r}_0 + (n\omega_0 z_0/\gamma^2)}{\mathbf{k} \cdot \mathbf{r}_0 - (n\omega_0 z_0\beta/c)} \right] J_n(\mathbf{k} \cdot \mathbf{r}_0 - n\omega_0 z_0\beta/c) \\ &\quad \cdot \delta(\omega - n\frac{\omega_0}{\gamma} - \mathbf{k} \cdot \mathbf{V}) \end{aligned} \quad (4-2-42)$$

$$\begin{aligned} J(\mathbf{k}, \omega) &= 2\pi q \sum_{n=-\infty}^{\infty} \left[ \frac{c\beta \mathbf{k} \cdot \mathbf{r}_0 + (n\omega_0 \mathbf{r}_0/\gamma)}{(\mathbf{k} - n\gamma\omega_0\beta/c) \cdot \mathbf{r}_0} \right] J_n[(\mathbf{k} - n\gamma\omega_0\beta/c) \cdot \mathbf{r}_0] \\ &\quad \cdot \delta(\omega - n\frac{\omega_0}{\gamma} - \mathbf{k} \cdot \mathbf{V}) \end{aligned} \quad (4-2-43)$$

Note that for  $\omega_0 z_0 \ll c$ , this equation reduces to (4-2-32). This is trivial for  $\mathbf{k} \cdot \mathbf{r}_0 \gg n\omega_0 z_0/c$ ; for  $\mathbf{k} \cdot \mathbf{r}_0 \approx \omega_0 z_0/c \ll 1$ , we have :

$$\begin{aligned} J_n(\mathbf{k} \cdot \mathbf{r}_0 - n\omega_0 z_0\beta/c) &\rightarrow 0 \text{ for } n \neq 0 ; \\ J_0 &\rightarrow 1 \quad \text{for } n = 0 \end{aligned} \quad (4-2-44)$$

Using these limits, both (4-2-32) and (4-2-43) tend to :

$$\begin{aligned} J(\mathbf{k}, \omega) &= 2\pi q V \delta(\omega - \mathbf{k} \cdot \mathbf{V}) + \pi q \left[ \frac{\omega_0}{\gamma} \mathbf{r}_0 + (\mathbf{k} \cdot \mathbf{r}_0) \mathbf{V} \right] \delta(\omega - \frac{\omega_0}{\gamma} - \mathbf{k} \cdot \mathbf{V}) \\ &\quad + \pi q \left[ \frac{\omega_0}{\gamma} \mathbf{r}_0 - (\mathbf{k} \cdot \mathbf{r}_0) \mathbf{V} \right] \delta(\omega + \frac{\omega_0}{\gamma} - \mathbf{k} \cdot \mathbf{V}) \end{aligned} \quad (4-2-45)$$

If the  $\mathbf{k}^\sigma$  of the point of stationary phase of the specific problem really satisfies  $\mathbf{k}^\sigma \cdot \mathbf{r}_0 \ll 1$ , (4-2-45) can also be used as an approximate expression for the current density. This is called the dipole approximation.

#### 4.3 Real field for separate terms $\delta(\omega \pm n\frac{\omega_0}{\gamma} - \mathbf{k} \cdot \mathbf{V})$

From the result in § 4-2, it is found that the Fourier



transform of the current density of a real source generally consists of pairs of terms of the form  $\delta(\omega + n\frac{\omega_0}{\gamma} - k \cdot V)$  and  $\delta(\omega - n\frac{\omega_0}{\gamma} - k \cdot V)$ . This two terms is not independent. Indeed, it can be shown that these two terms ( for one  $n$  ) will add up to be a real result. Let

$$J(k, \omega) = 2\pi \sum_{n=0}^{\infty} J_n^+(k) \delta(\omega - n\frac{\omega_0}{\gamma} - k \cdot V) + J_n^-(k) \delta(\omega + n\frac{\omega_0}{\gamma} - k \cdot V) \quad (4-3-1)$$

Since

$$J(k, \omega) = \frac{1}{(2\pi)^4} \int dx dt J(x, t) e^{-i(k \cdot x - \omega t)} \quad (4-3-2)$$

If  $J(x, t)$  is a real quantity, then

$$J^*(k, \omega) = J(-k, -\omega) \quad (4-3-3)$$

using this and (4-3-1) :

$$\begin{aligned} & \sum_n J_n^+(-k) \delta(\omega + n\frac{\omega_0}{\gamma} - k \cdot V) + J_n^-(-k) \delta(\omega - n\frac{\omega_0}{\gamma} - k \cdot V) \\ &= \sum_n J_n^{+\star}(k) \delta(\omega - n\frac{\omega_0}{\gamma} - k \cdot V) + J_n^{-\star}(k) \delta(\omega + n\frac{\omega_0}{\gamma} - k \cdot V) \end{aligned} \quad (4-3-4)$$

Since the  $\delta$ -function is discrete :

$$J_n^+(-k) = J_n^{-\star}(k) \quad \text{or} \quad J_n^{+\star}(k) = J_n^-(-k) \quad (4-3-5)$$

The results of (4-2-14) and (4-2-43) can be checked to be consistent with this equation if we note the formula of Bessel functions :

$$J_{-n}(z) = J_n(-z) \quad (4-3-6)$$

Apply (2-13) and (4-3-1) :

$$\begin{aligned} E(x, t) = \sum_{n=0}^{\infty} \int \frac{-i dk}{2\pi^2} & \left\{ \left[ \frac{\omega \bar{\Gamma}(k, \omega)}{\mathcal{D}(k, \omega)} \right]_{D+n} J_n^+(k) e^{i[k \cdot (x - Vt) - n\frac{\omega_0}{\gamma} t]} \right. \\ & \left. + \left[ \frac{\omega \bar{\Gamma}(k, \omega)}{\mathcal{D}(k, \omega)} \right]_{D-n} J_n^-(k) e^{i[k \cdot (x - Vt) + n\frac{\omega_0}{\gamma} t]} \right\} \end{aligned} \quad (4-3-7)$$

where  $D_{+n}$  means  $\omega \rightarrow n\frac{\omega_0}{\gamma} + k \cdot V$

$D_{-n}$  means  $\omega \rightarrow -n\frac{\omega_0}{\gamma} + k \cdot V$

Now call the  $k$  in the second term  $-k$ , and make use of the property of the dielectric tensor of the medium (e.g. Landau, 1960):

$$\bar{\epsilon}(-k, -\omega) = \bar{\epsilon}^*(k, \omega) \quad (4-3-8)$$

Then, by the definition of  $\mathcal{D}$ :

$$\begin{aligned} \mathcal{D}(-k, -\omega) &= \det |\bar{\Delta}(-k, -\omega)| = \det |\bar{\Delta}^*(k, \omega)| \\ &= \mathcal{D}^*(k, \omega) \end{aligned} \quad (4-3-9)$$

Note that if  $\bar{\epsilon}$  is Hermitian, i.e.

$$\bar{\epsilon}(k, \omega) = \bar{\epsilon}^{*T}(k, \omega) \quad (4-3-10)$$

then  $\mathcal{D}(k, \omega)$  is real.

$$\begin{aligned} \bar{\Gamma}(-k, -\omega) &= \mathcal{D}(-k, -\omega) \bar{\Delta}^{-1}(-k, -\omega) = \mathcal{D}^*(k, \omega) \bar{\Delta}^{*-1}(k, \omega) \\ &= \bar{\Gamma}^*(k, \omega) \end{aligned} \quad (4-3-11)$$

So, (4-3-7) becomes:

$$E(x, t) = 2\Re e \sum_{n=0}^{\infty} \int \frac{-i dk}{2\pi^2} \left[ \frac{\omega \bar{\Gamma}(k, \omega)}{\mathcal{D}(k, \omega)} \right]_{D+n} \cdot J_n^+(k) e^{i[k \cdot (x - Vt) - n \frac{\omega_0}{\gamma} t]} \quad (4-3-12)$$

So, a pair of terms  $\delta(\omega \pm n \frac{\omega_0}{\gamma} - k \cdot V)$  will give a real contribution to the E field. We can then just pick up one term  $\delta(\omega - n \frac{\omega_0}{\gamma} - k \cdot V)$  in the calculation and twice the real part as the final result.



## CHAPTER V

### RADIATION IN VACUUM

#### 5.1 Moving dipole radiation by usual method

The radiation field of a stationary dipole  $p_o e^{-i\omega_o t}$  in vacuum is well known (e.g. Jackson, 1975, eqn. 9.18):

$$E(x,t) = - \frac{k \times (k \times p_o)}{r} e^{i(k \cdot x - \omega_o t)} \quad (5-1-1)$$

$$B(x,t) = \hat{k} \times E(x,t) \quad (5-1-2)$$

where

$$r \equiv |x|, \quad k = \omega_o \hat{x} / c \quad (5-1-3)$$

Then, the radiation field of this dipole moving in constant velocity  $V$  can be found by Lorentz transformation to a frame moving in  $-V$  relative to the source :

$$E(x,t) = - \frac{\gamma k \times [(k - k\beta) \times p]}{\rho [1 + \gamma^2 \tau^2]^{1/2}} e^{i(k \cdot x - \omega t)} \quad (5-1-4)$$

where

$$p \equiv p_{o\perp} + p_{oz} / \gamma \hat{z} \quad (5-1-5)$$

$$x = \rho (\cos\phi \hat{x} + \sin\phi \hat{y}) + z \hat{z} \quad (5-1-6)$$

$$k = k_\rho (\cos\phi \hat{x} + \sin\phi \hat{y}) + k_z \hat{z} \quad (5-1-7)$$

$$k_\rho = \omega_o / c [1 + \gamma^2 \tau^2]^{1/2} \quad (5-1-8)$$

$$k_z = \frac{\gamma \omega_o}{c} \left[ \beta - \frac{\gamma \tau}{[1 + \gamma^2 \tau^2]^{1/2}} \right] \quad (5-1-9)$$

$$\omega = \gamma \omega_o \left[ 1 - \frac{\gamma \beta \tau}{[1 + \gamma^2 \tau^2]^{1/2}} \right] \quad (5-1-10)$$

$$\tau \equiv (Vt - z) / \rho \quad (5-1-11)$$

The magnetic field is still given by (5-1-2) but with the quantities  $k, \omega$  depends on space-time now. Note that if we put  $V = 0$ , (5-1-4) reduces to (5-1-1) as expected.

For a linearly oscillating charge given by (4-2-27) :

$$r(t) \cong r_0 e^{-i\omega_0 t/\gamma} + vt \hat{z} \cong vt \quad (5-1-12)$$

the radiation field can also be calculated by Liénard-Wiechart's method :

$$E(x,t) = \frac{q}{c} \left[ \frac{n \times (n - \vec{\beta}) \times \dot{\vec{\beta}}}{(1 - \vec{\beta} \cdot n)^3 R} \right]_{t' = t - \frac{|x - r(t')|}{c}} \quad (5-1-13)$$

where

$$R = Rn \equiv x - r(t)$$

$$c\vec{\beta} = \dot{r} = r_0 \frac{\omega_0}{\gamma} e^{-i\omega_0 t/\gamma} + v \hat{z} \cong v \quad (5-1-14)$$

$$c\dot{\vec{\beta}} = \dot{r}' = -i\left(\frac{\omega_0}{\gamma}\right)^2 r_0 e^{-i\omega_0 t/\gamma} \quad (5-1-15)$$

The retarded time  $t'$  can thus be found as :

$$t' = \gamma^2 \left( t - \frac{\beta z}{c} \right) - \frac{\gamma \rho}{c} [1 + \gamma^2 \tau^2]^{1/2} \quad (5-1-16)$$

where  $\tau$  is still given by (5-1-11). Introduce the quantities  $k, \omega$  :

$$\omega = \frac{\omega_0}{\gamma} + k \cdot v \quad (5-1-17)$$

$$k = \frac{\omega(x - vt')}{c|x - vt'|} = k_\rho \hat{\rho} + k_z \hat{z} \quad (5-1-18)$$

such that the phase factor can be shown to be :

$$\frac{\omega_0}{\gamma} t' = \omega t - k \cdot x \quad (5-1-19)$$

By this and (5-1-17) and (5-1-18),  $\omega$  and  $k$  can be expressed in  $\tau$ , which are also by (5-1-8)-(5-1-10). Substituting  $n = \hat{k}$ ,

$$R = c(t - t') \cong c\rho k [1 + \gamma^2 \tau^2]^{1/2} / \omega_0, \quad (5-1-20)$$

$$1 - \vec{\beta} \cdot n = (k - \vec{\beta} \cdot k)/k \cong \omega_0 / \gamma ck, \quad (5-1-21)$$

and (5-1-14), (5-1-15), into (5-1-13) :

$$E(x,t) = - \frac{i\gamma k \times [(k - k\vec{\beta}) \times qr_0]}{\rho [1 + \gamma^2 \tau^2]^{1/2}} e^{i(k \cdot x - \omega t)} \quad (5-1-22)$$

which is the same as (5-1-4) if we identify :

$$p = iqr_0 \quad (5-1-23)$$

The B field is still given by (5-1-2).



## 5.2 Using Lai and Chan's method

For vacuum,  $\bar{\epsilon} = \bar{I}$ , so by (2-9) and (2-11) :

$$\bar{\Delta}(k, \omega) = (\omega^2 - c^2 k^2) \bar{I} + c^2 k k \quad (5-2-1)$$

$$D(k, \omega) = \omega^2 (\omega^2 - c^2 k^2)^2 \quad (5-2-2)$$

$$\bar{\Gamma}(k, \omega) = (\omega^2 - c^2 k^2) (\omega^2 \bar{I} - c^2 k k) \quad (5-2-3)$$

The DWS for a proper frequency  $\omega_0$  is given by (3-2-2). From its shape (Fig. 3-4), it can be seen easily that there is only one stationary phase point  $k^\sigma$  for one observation point  $(x, t)$ . Since the DWS is an ellipsoid, i.e. (3-2-2) satisfies the form of (3-4-1),  $k, \omega$  can be found as a function of  $\tau$  immediately by using (3-4-4) - (3-4-8). The results are just the same as (5-1-7) - (5-1-11). Curvatures of the surface can also be found by (3-4-10) - (3-4-12) :

$$\lambda_{kz} = - \frac{c}{\gamma^2 \omega_0} \left[ \frac{1 + \gamma^2 \tau^2}{1 + \tau^2} \right]^{3/2} \quad (5-2-4)$$

$$\lambda_\phi = - \frac{c}{\omega_0} \left[ \frac{1 + \gamma^2 \tau^2}{1 + \tau^2} \right]^{1/2} \quad (5-2-5)$$

$$K = \left[ \frac{c}{\gamma \omega_0} \frac{1 + \gamma^2 \tau^2}{1 + \tau^2} \right]^2 \quad (5-2-6)$$

The far electric field is given by (2-20). Since both  $\lambda_{kz}$  and  $\lambda_\phi$  are smaller than zero, by (2-22) :

$$A^\sigma = -i \quad (5-2-7)$$

By (5-2-2) :

$$\frac{\partial D}{\partial k} = -4c^2 \omega^2 (\omega^2 - c^2 k^2) k \quad (5-2-8)$$

$$\frac{\partial D}{\partial \omega} = 2\omega (\omega^2 - c^2 k^2) (3\omega^2 - c^2 k^2) \quad (5-2-9)$$

since  $\frac{\partial D}{\partial \omega} > 0$  for  $\omega^2 = c^2 k^2$ , so using (2-19) :

$$\left( \frac{\partial D}{\partial k} + \frac{\partial D}{\partial \omega} \mathbf{v} \right)_N = -4c\omega^2 \omega_0 (\omega^2 - c^2 k^2) \left[ \frac{1 + \tau^2}{1 + \gamma^2 \tau^2} \right]^{1/2} \quad (5-2-10)$$

Also,

$$|x - vt| = \rho \sqrt{1 + \tau^2} \quad (5-2-11)$$

Using the linearly oscillating charge as the moving radiating source, and taking the dipole approximation, i.e. using

(4-2-45) :

$$J(k) = \frac{q}{2} \left[ \frac{\omega_0 r_0}{\gamma} + (k \cdot r_0) V \right] \quad (5-2-12)$$

Substitute above equations into (2-20), finally we get :

$$E(x, t) = \frac{4\gamma q (k^2 \bar{I} - k k) \cdot \left[ \frac{\omega_0 r_0}{\gamma} + (k \cdot r_0) V \right]}{\rho [1 + \gamma^2 \tau^2]^{1/2}} e^{i(k \cdot x - \omega t)} \quad (5-2-13)$$

where an additional factor of 4 has been multiplied since there are two degenerate mode of  $\omega^2 = c^2 k^2$  in (5-2-2), and that there is the  $n = -1$  term in the current density (4-2-31). By direct comparison, it can be shown that :

$$\begin{aligned} \frac{1}{\omega} (k^2 \bar{I} - k k) \cdot \left[ \frac{\omega_0 r_0}{\gamma} + (k \cdot r_0) V \right] \\ = [k(k - k\beta) \bar{I} - (k - k\beta)k] \cdot r_0 \end{aligned} \quad (5-2-14)$$

$$= -k \times [(k - k\beta)k \times r_0] \quad (5-2-15)$$

So, the result (5-1-22) is again reproduced. This gives a check to Lai and Chan's formulation.

It is easy to extend the above calculation to the exact current density (4-2-43) :

$$E = \sum_{n=-\infty}^{\infty} \frac{-i\gamma J_n [(k_n - \gamma n \frac{\omega_0}{c} \beta) \cdot r_0] k_n \times [(k_n - k_n \beta) k_n \times r_0]}{2[k_n - (\gamma n \omega_0 \beta / c)] \cdot r_0 \rho [1 + \gamma^2 \tau^2]^{1/2}} e^{i[k_n \cdot x - \omega_n t]} \quad (5-2-16)$$

where  $k_n, \omega_n$  are given by (5-1-16) - (5-1-11) with  $\omega_0 \rightarrow n\omega_0$ .

Similar calculation for the helically moving charge can also be done by replacing the current density  $J(k)$  by (4-2-14).

The above result can be checked to be relativistic in the sense that if we put  $V = 0$  in the  $E$  and  $B$  field and then use the Lorentz transformation to transform it to a frame moving in  $-V$ , the same  $E$  and  $B$  field can be reproduced. This can also be seen from the calculation from (5-1-1) to (5-1-4).



### 5.3 Radiation energy

The Poyting vector for the Moving dipole radiation can be found by using (2-6), or simply by using :

$$\begin{aligned}\langle S(x,t) \rangle &= \frac{c}{16\pi} [E(x,t) \times B(x,t) + c.c.] \\ &= \frac{c^2 k}{8\pi\omega} |E|^2\end{aligned}\quad (5-3-1)$$

with  $E(x,t)$  given by (5-2-13). Both two method give :

$$\langle S \rangle = \frac{c^2}{8\pi} \left( \frac{\gamma q}{\rho} \right)^2 \frac{|(k^2 \bar{I} - kk) \cdot [\frac{\omega_0 r_0}{\gamma} + (k \cdot r_0) V]|^2}{(1 + \gamma^2 \tau^2) \omega^3} k \quad (5-3-2)$$

define :

$$R \equiv r_0 + \frac{\gamma}{\omega_0} (k \cdot r_0) V \quad (5-3-3)$$

then,

$$\langle S \rangle = \frac{q^2 \omega_0^2 [k^2 R^2 - (k \cdot R)^2]}{8\pi\omega\rho^2(1 + \gamma^2 \tau^2)} k \quad (5-3-4)$$

By (2-29), the power received per unit solid angle subtended at the retarded position is :

$$\frac{dP}{d\Omega} = |x - Vt'|^2 |\langle S \rangle|$$

where

$$\begin{aligned}|x - Vt'|^2 &= (x - Vt)^2 \frac{V_E^2}{(V_E - V)^2} \\ &= \left( \frac{\omega}{\omega_0} \right)^2 \rho^2 (1 + \tau^2)\end{aligned}\quad (5-3-5)$$

So,

$$\frac{dP}{d\Omega} = \frac{q^2 [k^2 R^2 - (k \cdot R)^2] \omega k}{8\pi} \quad (5-3-6)$$

The emitted power per unit solid angle is then found by (2-30) with  $f$  found by (2-31) :

$$f = 1 - \beta \cos \theta \quad (5-3-7)$$

$$\frac{dP'}{d\Omega} = \frac{q^2 (1 - \beta \cos \theta) [k^2 R^2 - (k \cdot R)^2] \omega k}{8\pi} \quad (5-3-9)$$

where  $k$ ,  $\omega$  are now expressed as functions of  $\theta$  (the angle between  $V_E$  and  $V$  and note the  $V_E = c\hat{k}$ ) by (3-2-3) :

$$ck = \omega = \omega_0 / \gamma(1 - \beta \cos \theta) \quad (5-3-8)$$

The total radiation power can be found by integration :

$$W = \int \frac{dP}{d\Omega} \sin \theta \, d\theta \, d\phi \quad (5-3-10)$$

In order to simplify the calculation of this, we choose

$$\mathbf{r}_0 = x_0 \hat{x} + \frac{z_0}{\gamma} \hat{z}$$

without loss of generality, then,

$$\mathbf{k} \cdot \mathbf{r}_0 = k [x_0 \sin \theta \cos \phi + \frac{z_0}{\gamma} \cos \theta] \quad (5-3-11)$$

$$(\mathbf{k} \cdot \mathbf{R})^2 = (\mathbf{k} \cdot \mathbf{r}_0)^2 (1 + \frac{\gamma \mathbf{k} \cdot \mathbf{V}}{\omega_0})^2 = (\frac{\gamma \omega}{\omega_0})^2 (\mathbf{k} \cdot \mathbf{r}_0)^2 \quad (5-3-12)$$

$$R^2 = r_0^2 + 2 \frac{z_0}{\omega_0} V (\mathbf{k} \cdot \mathbf{r}_0) + (\frac{\gamma}{\omega_0})^2 V^2 (\mathbf{k} \cdot \mathbf{r}_0)^2 \quad (5-3-13)$$

Substituting these equations into (5-3-9), (5-3-10) and using a change of variable  $y \equiv \cos \theta$ , we get,

$$W = \frac{q^2 \omega_0^2}{8c^3 \gamma^4} \int_{-1}^1 dy \left\{ \frac{2r_0^2}{(1-\beta y)^3} + \frac{4z_0^2 \beta y}{\gamma^2 (1-\beta y)^4} - \frac{[x_0^2 (1-y^2) + 2(\frac{z_0}{\gamma})^2 y^2]}{\gamma^2 (1-\beta y)^5} \right\}$$

Using

$$\int_{-1}^1 \frac{dy}{(1-\beta y)^3} = 2\gamma^4 \quad (5-3-14)$$

$$\int_{-1}^1 \frac{y dy}{(1-\beta y)^4} = \frac{8\beta \gamma^6}{3} \quad (5-3-15)$$

$$\int_{-1}^1 \frac{dy}{(1-\beta y)^5} = 2\gamma^8 (1 + \beta^2) \quad (5-3-16)$$

$$\int_{-1}^1 \frac{y^2 dy}{(1-\beta y)^5} = 2\gamma^8 (1 + \beta^2) - \frac{4}{3} \gamma^6 \quad (5-3-17)$$

Finally we get,

$$W = \frac{q^2 \omega_0^4}{3c^3} r_0^2 \quad (5-3-18)$$

Note that  $r_0^2 = x_0^2 + (\frac{z_0}{\gamma})^2$ .

For  $\beta = 0$ , this result is the same as the well known result (e.g. Jackson, 1975, eqn 9-24)

The above discussion shows the essential procedure for the calculation of radiation power of a moving source.



## CHAPTER VI

### RADIATION IN ISOTROPIC NON-DISPERSIVE MEDIA

We now consider the next simple media in which the radiation from a uniform moving source can be easily found, i.e. the isotropic non-dispersive media with a scalar dielectric constant. The DWS was found in § 3.2.2. The essential difference between this case and vacuum is that when the velocity of the source exceeds the speed of light in the medium, i.e.  $V > c/\sqrt{\epsilon}$ , the DWS changes from an ellipsoid to a hyperboloid so that Cerenkov radiation may occur.

#### 6.1 $V < c / \sqrt{\epsilon}$

For an isotropic non-dispersive medium :

$$\bar{\epsilon} = \epsilon \bar{I} \quad (6-1-1)$$

$$\bar{\Delta}(k, \omega) = (\epsilon \omega^2 - c^2 k^2) \bar{I} + c^2 k k \quad (6-1-2)$$

$$\bar{D}(k, \omega) = \epsilon \omega^2 (\epsilon \omega^2 - c^2 k^2)^2 \quad (6-1-3)$$

$$\bar{\Gamma}(k, \omega) = (\epsilon \omega^2 - c^2 k^2) (\epsilon \omega^2 \bar{I} - c^2 k k) \quad (6-1-4)$$

The DWS for the mode  $\epsilon \omega^2 - c^2 k^2 = 0$  is given by (3-2-5) which is of the form (3-4-1). For  $\beta < 1/\sqrt{\epsilon}$ , the DWS is an ellipsoid and so there is only one point of stationary phase for a given  $(x, t)$  and the quantities at the corresponding point of stationary phase can be expressed as functions of  $\tau$  using (3-4-4) - (3-4-8):

$$k_{\rho} = \frac{\gamma_{\epsilon} \omega_0}{\gamma c} \left[ \frac{\epsilon}{1 + \gamma_{\epsilon}^2 \tau^2} \right]^{1/2} \quad (6-1-5)$$

$$k_z = \frac{\gamma_{\epsilon}^2 \omega_0}{\gamma c} \left[ \epsilon \beta - \frac{\gamma_{\epsilon} \tau \sqrt{\epsilon}}{[1 + \gamma_{\epsilon}^2 \tau^2]^{1/2}} \right] \quad (6-1-6)$$

$$\omega = \frac{\gamma_{\epsilon}^2 \omega_0}{\gamma} \left[ 1 - \frac{\gamma_{\epsilon} \beta \tau \sqrt{\epsilon}}{[1 + \gamma_{\epsilon}^2 \tau^2]^{1/2}} \right] \quad (6-1-7)$$

$$k = \frac{\gamma_\epsilon^2 \omega_o}{\gamma c} \left[ \sqrt{\epsilon} - \frac{\gamma_\epsilon \beta \tau \epsilon}{[1 + \gamma_\epsilon^2 \tau^2]^{1/2}} \right] \quad (6-1-8)$$

where

$$\gamma_\epsilon^2 \equiv 1 / (1 - \epsilon \beta^2) \quad (6-1-9)$$

Note that for  $\epsilon = 1$ , (6-1-5) to (6-1-9) reduce to the expressions (5-1-8) - (5-1-10) in the vacuum case, since  $\gamma_\epsilon = \gamma$  in this case. We see that the space-time dependence of  $k, \omega$  is similar to that of vacuum and is single value, so the radiation field should be of the similar form. The curvatures of DWS at the point of stationary phase are:

$$\lambda_{kz} = - \frac{c \gamma}{\omega_o \gamma_\epsilon^3 \sqrt{\epsilon}} \left[ \frac{1 + \gamma_\epsilon^2 \tau^2}{1 + \tau^2} \right]^{3/2} \quad (6-1-10)$$

$$\lambda_\phi = - \frac{c \gamma}{\omega_o \gamma_\epsilon \sqrt{\epsilon}} \left[ \frac{1 + \gamma_\epsilon^2 \tau^2}{1 + \tau^2} \right]^{1/2} \quad (6-1-11)$$

$$K = \frac{1}{\epsilon} \left[ \frac{c \gamma}{\omega_o \gamma_\epsilon^2} \frac{1 + \gamma_\epsilon^2 \tau^2}{1 + \tau^2} \right]^2 \quad (6-1-12)$$

By (6-1-3) :

$$\frac{\partial D}{\partial k} = - 4c^2 \epsilon \omega^2 (\epsilon \omega^2 - c^2 k^2) k \quad (6-1-13)$$

$$\frac{\partial D}{\partial \omega} = 2\epsilon \omega (\epsilon \omega^2 - c^2 k^2) (3\epsilon \omega^2 - c^2 k^2) \quad (6-1-14)$$

$$\left( \frac{\partial D}{\partial k} + \frac{\partial D}{\partial \omega} V \right)_N = -4c\omega^2 \omega_o \epsilon \frac{\gamma_\epsilon}{\gamma} (\epsilon \omega^2 - c^2 k^2) \left[ \frac{(1 + \tau^2) \epsilon}{1 + \gamma_\epsilon^2 \tau^2} \right]^{\frac{1}{2}} \quad (6-1-15)$$

we use the  $J(k)$  in (4-2-45) as in the vacuum case, then

$$E(x, t) = \frac{i\gamma_\epsilon q (k^2 \bar{I} - kk) \cdot \left[ \frac{\omega_o r_o}{\gamma} + (k \cdot r_o) V \right]}{\epsilon \rho [1 + \gamma_\epsilon^2 \tau^2]^{1/2}} e^{i(k \cdot x - \omega t)} \quad (6-1-16)$$

We can make use of the following relation to rewrite the result :

$$\begin{aligned} (k^2 \bar{I} - kk) \cdot \left[ \frac{\omega_o r_o}{\gamma} + (k \cdot r_o) V \right] / \omega \\ = -k \times [(k - ck^2 \beta / \omega) \times r_o] \end{aligned} \quad (6-1-17)$$

So,



$$E(x,t) = -\gamma_\epsilon \frac{iqk \times [(k - \sqrt{\epsilon} k\beta) \times r_o]}{\epsilon\rho [1 + \gamma_\epsilon^2 \tau^2]^{1/2}} e^{i(k \cdot x - \omega t)} \quad (6-1-18)$$

and the magnetic field is :

$$B(x,t) = \sqrt{\epsilon} \hat{k} \times E(x,t) \quad (6-1-19)$$

Note that if we put  $\epsilon = 1$ , the vacuum dipole field (5-1-22) is again reproduced.

For  $\beta = 0$ , (6-1-18) gives the well-known radiation field for stationary dipole (e.g. Chen, 1983, eqn.9-65).

The time-averaged Poynting vector can then be found as :

$$\langle S \rangle = \frac{\gamma_\epsilon^2 \omega_o^2 q^2 [k^2 R^2 - (k \cdot R)^2] k}{8\pi \gamma^2 \epsilon \omega \rho^2 (1 + \gamma_\epsilon^2 \tau^2)} \quad (6-1-20)$$

where  $R$  has been defined by (5-3-3).

The power received per unit solid angle is :

$$\frac{dP}{d\Omega} = \frac{q^2 [k^2 R^2 - (k \cdot R)^2] \omega^2}{8\pi c \sqrt{\epsilon}} \quad (6-1-21)$$

The emitted power per unit solid angle is :

$$\frac{dP'}{d\Omega} = \frac{q^2 [k^2 R^2 - (k \cdot R)^2] \omega^2}{8\pi c \sqrt{\epsilon}} (1 - \beta \sqrt{\epsilon} \cos\theta) \quad (6-1-22)$$

with now  $\omega$ ,  $k$  are functions of  $\theta$  given by (3-3-6) :

$$ck = \omega \sqrt{\epsilon} = \sqrt{\epsilon} \omega_o / \gamma (1 - \beta \sqrt{\epsilon} \cos\theta) \quad (6-1-23)$$

The total radiation power can be found by integration of (6-1-22) using similar method as § 5.3.

## 6.2 $v > c / \sqrt{\epsilon}$

The DWS in this case is no longer ellipsoid, but is hyperboloid. In general, there should be two points of stationary phase for each observation space time  $(x,t)$ . So, (6-1-5) - (6-1-8) should be changed to:

$$k = \sigma k_\rho \hat{\rho} + k_z \hat{z} \quad (6-2-1)$$

$$k_{\rho} = \frac{\gamma_{\epsilon} \omega_0}{\gamma c} \left[ \frac{\epsilon}{-1 + \gamma_{\epsilon}^2 \tau^2} \right]^{1/2} \quad (6-2-2)$$

$$k_z = \frac{\gamma_{\epsilon}^2 \omega_0}{\gamma c} \left[ -\epsilon \beta + \frac{\sigma \gamma_{\epsilon} \tau \sqrt{\epsilon}}{[-1 + \gamma_{\epsilon}^2 \tau^2]^{1/2}} \right] \quad (6-2-3)$$

$$\omega = \frac{\gamma_{\epsilon}^2 \omega_0}{\gamma} \left[ -1 + \frac{\sigma \gamma_{\epsilon} \beta \tau \sqrt{\epsilon}}{[-1 + \gamma_{\epsilon}^2 \tau^2]^{1/2}} \right] \quad (6-2-4)$$

$$k = \frac{\omega \sqrt{\epsilon}}{c} = \frac{\gamma_{\epsilon}^2 \omega_0}{\gamma c} \left[ -\sqrt{\epsilon} + \frac{\sigma \gamma_{\epsilon} \beta \tau \epsilon}{[-1 + \gamma_{\epsilon}^2 \tau^2]^{1/2}} \right] \quad (6-2-5)$$

where

$$\gamma_{\epsilon}^2 \equiv 1 / |1 - \epsilon \beta^2| ; \sigma = 1, -1$$

The two  $\omega$  in (6-2-4) are one + value and one - value, see Fig. 6-1. So,  $k$  in (6-2-5) also have  $\pm$  value.

Note that if  $\gamma_{\epsilon}^2 \tau^2 - 1 < 0$ , the above equations become complex which actually means that there is no radiation field in this case. To see this, consider  $t = 0$ , so  $\tau = -z / \rho$ . (see Fig. 6-1), if  $\gamma_{\epsilon}^2 \tau^2 - 1 > 0$ , then

$$\tan^2 \theta \equiv (z/\rho)^2 \geq \epsilon \beta^2 - 1 \quad (6-2-6)$$

The cone constructed by  $\tan \theta = 1/\gamma_{\epsilon}^2$  is just the Cerenkov cone, since

$$\cos \theta = \frac{c}{V \sqrt{\epsilon}} = \frac{\text{velocity of light in the medium}}{\text{velocity of the radiation source}} \quad (6-2-7)$$

This means that the reason for the absence of radiation at some region is just because the radiation propagate slower than the source. So, from this physical consideration, radiation <sup>exists inside</sup> ~~only~~ the backward Cerenkov cone, i.e. behind the source :

$$(z/\rho) < -(\epsilon \beta^2 - 1)^{1/2} = -1 / \gamma_{\epsilon} \quad (6-2-8)$$

Also, the reason for the existence of two points of stationary phase is due the well-known fact that there exist two retarded source positions for the velocity of the source larger than the velocity of light of that medium.



$$\lambda_{kz} = - \sigma \frac{c \gamma}{\omega_o \gamma_\epsilon^3 \sqrt{\epsilon}} \left[ \frac{-1 + \gamma_\epsilon^2 \tau^2}{1 + \tau^2} \right]^{3/2} \quad (6-2-9)$$

$$\lambda_\phi = - \sigma \frac{c \gamma}{\omega_o \gamma_\epsilon \sqrt{\epsilon}} \left[ \frac{-1 + \gamma_\epsilon^2 \tau^2}{1 + \tau^2} \right]^{1/2} \quad (6-2-10)$$

$$K = \frac{1}{\epsilon} \left[ \frac{c \gamma}{\omega_o \gamma_\epsilon^2} \frac{-1 + \gamma_\epsilon^2 \tau^2}{1 + \tau^2} \right]^2 \quad (6-2-11)$$

So, by (2-22) :

$$A^\sigma = -i\sigma \quad (6-2-12)$$

$\partial D / \partial k$  and  $\partial D / \partial \omega$  are still given by (6-1-13) and (6-1-14), but now note that  $\partial D / \partial \omega$  is smaller than zero. So, similar to (6-1-15) :

$$\begin{aligned} \left( \frac{\partial D}{\partial k} + \frac{\partial D}{\partial \omega} v \right)_N \\ = -4\sigma c \omega^2 \omega_o \epsilon \frac{\gamma_\epsilon}{\gamma} (\epsilon \omega^2 - c^2 k^2) \left[ \frac{(1 + \tau^2) \epsilon}{-1 + \gamma_\epsilon^2 \tau^2} \right]^{1/2} \end{aligned} \quad (6-2-13)$$

Using the same  $J(k)$  as the above section, we have :

$$E = \sum_{\sigma} - \frac{i \gamma_\epsilon q k_\sigma \times [(k_\sigma - \sqrt{\epsilon} k_\sigma \beta) \times r_o]}{\epsilon \rho [-1 + \gamma_\epsilon^2 \tau^2]^{2 \cdot 1/2}} e^{i(k_\sigma \cdot x - \omega_\sigma t)} \quad (6-2-14)$$

And the magnetic field is :

$$B(x, t) = \sum_{\sigma} \sqrt{\epsilon} \hat{k}_\sigma \times E_\sigma(x, t) \quad (6-2-15)$$

Note that (6-2-14) diverges at  $\gamma_\epsilon^2 \tau^2 = 1$ . This is because  $\omega \rightarrow \infty$  on this surface, so the non-dispersive model is no-longer valid.

The time-average Poynting vector can also be found :

$$\langle S \rangle = \sum_{\sigma} \frac{\gamma_\epsilon^2 \omega_o^2 q^2 [k_\sigma^2 R_\sigma^2 - (k_\sigma \cdot R_\sigma^2)] k_\sigma}{8\pi \gamma^2 \epsilon \omega_o \rho^2 (-1 + \gamma_\epsilon^2 \tau^2)} \quad (6-2-16)$$

Note that the interference between the  $\sigma = 1$  and  $\sigma = -1$  waves is not important in the time-average Poynting vector since the frequencies of the two waves are different in general and thus the interference term vanishes after taking time

average. The received and emitted power per unit solid angle are still given by (6-1-21) and (6-1-22) for each  $\alpha$ .

### 6.3 Cerenkov radiation from the $\delta(\omega - \mathbf{k} \cdot \mathbf{V})$ term

The contribution of the term  $\delta(\omega - \mathbf{k} \cdot \mathbf{V})$  can be seen qualitatively by putting  $\omega_0 \rightarrow 0$  in (6-1-19) and (6-2-14). For  $\beta < 1/\sqrt{\epsilon}$ , its contribution is zero. This means that a uniformly moving charge in an isotropic non-dispersive medium with velocity smaller than the velocity of light of the medium does not radiate. This does not mean that the E field is zero in this case. Instead, by direct integration of (2-10), the E field of this charge can be found as :

$$E(\mathbf{x}, t) = \frac{\gamma_{\epsilon} q (\mathbf{x} - \mathbf{V}t)}{\epsilon [\gamma_{\epsilon}^2 (z - Vt)^2 + \rho^2]^{3/2}} \quad (6-3-1)$$

By putting  $\epsilon = 1$ , i.e.  $\gamma_{\epsilon} = \gamma$ , (6-3-1) is just the well-known expression of the electric field of a uniformly moving charge in vacuum (e.g. Jackson, 1975, eqn. 11.152).

For  $\beta > 1/\sqrt{\epsilon}$ , (6-2-14) suggests that the radiation only exist in the divergent cone  $\gamma_{\epsilon}^2 \tau^2 = 1$ . This is just the Cerenkov radiation. From now on we just consider this case.

If we put  $\omega_0 = 0$  into (3-2-5), we find that the DWS for Cerenkov radiation is :

$$k_{\rho}^2 = k_z^2 / \gamma_{\epsilon}^2 \quad (6-3-2)$$

which is a conical surface (Fig 6-3). So,

$$k'_{\rho} = \tau = 1 / \gamma_{\epsilon} \quad (6-3-3)$$

$$k''_{\rho} = 0 \quad (6-3-4)$$

This means that for a fix  $\phi$ , all  $k$  on the DWS are the point of stationary phase for the observation point satisfying  $\tau = 1 / \gamma_{\epsilon}$ . The curvatures can be found by (3-3-8) - (3-3-10) :



$$\lambda_\phi = -\gamma_\epsilon / k_z (1 + \tau^2)^{1/2} \quad (6-3-5)$$

$$\lambda_{kz} = K = 0 \quad (6-3-6)$$

Therefore, (2-20) can not be used here since  $K = 0$  has been assumed in (2-20). Instead, we should start from (2-16) and expanding the phase factor as :

$$k \cdot (x - vt) \cong [k^\sigma + \frac{1}{2} \frac{\partial^2 k}{\partial k_z^2} (k_z - k_z^\sigma)^2 + \frac{1}{2} \frac{\partial^2 k}{\partial \phi^2} (\phi - \phi^\sigma)^2] \cdot (x - vt) \quad (6-3-7)$$

where the stationary phase point condition has been used :

$$\frac{\partial k}{\partial k_z} \cdot (x - vt) = \frac{\partial k}{\partial \phi} \cdot (x - vt) = 0 \quad (6-3-8)$$

using similar derivation of (2-20), we get :

$$E(x, t) = \frac{q}{\pi} \int_{-\infty}^{\infty} \left| \frac{\partial k}{\partial k_z} \right| dk_z \left[ \frac{2\pi}{|\lambda_\phi| |x - vt|} \right]^{1/2} e^{i \pi \text{sign}(\lambda_\phi)/4} \cdot \left[ \frac{\omega \bar{\Gamma}(k, \omega)}{c \frac{\partial D}{\partial k} + \frac{\partial D}{\partial \omega} v_N} \right]_D \cdot v e^{ik \cdot (x - vt)} \quad (6-3-9)$$

By (3-3-2) :

$$\left| \frac{\partial k}{\partial k_z} \right| = | \tau \hat{\rho} + \hat{z} | = (1 + \tau^2)^{1/2} \quad (6-3-9)$$

By (6-3-5) :

$$\text{sign}(\lambda_\phi) = -\text{sign}(k_z) \quad (6-3-10)$$

and since  $\hat{N}$  perpendicular to  $k^\sigma$  :

$$\exp i k^\sigma \cdot (x - vt) = 1 \quad (6-3-11)$$

$\bar{\Gamma}$ ,  $\partial D / \partial k$  and  $\partial D / \partial \omega$  are still given by (6-1-2), (6-1-13) and (6-1-14). Since  $\partial D / \partial \omega$  larger than zero for  $\omega > 0$  and smaller than zero for  $\omega < 0$  then :

$$\left( \frac{\partial D}{\partial k} + \frac{\partial D}{\partial \omega} v \right)_N = -4 \text{sign}(k_z) \omega^2 \epsilon (\epsilon \omega^2 - c^2 k^2) c^2 k_\rho (1 + \tau^2)^{1/2} \quad (6-3-12)$$

Since on the DWS ,

$$k = \text{sign}(k_z) k_\rho \hat{\rho} + k_z \hat{z} \quad (6-3-13)$$

and using (6-1-17) :

$$\begin{aligned} (k^2 \bar{I} - k k) \cdot v &= -k \times (k \times v) \\ &= \text{sign}(k_z) k_\rho k_z (\hat{\rho} - \hat{z} / \gamma_\epsilon) v \end{aligned} \quad (6-3-14)$$

substitute the above equations into (6-3-9) :

$$\begin{aligned}
 E(x,t) &= \frac{q}{4\epsilon} \left( \frac{2}{\pi \gamma_{\epsilon} \rho} \right)^{1/2} (\hat{\rho} - \hat{z}/\gamma_{\epsilon}) \int_{-\infty}^{\infty} \sqrt{|k_z|} e^{i\pi \text{sign}(k_z)/4} dk_z \\
 &= \frac{\beta q}{2} \left( \frac{1}{\pi \gamma_{\epsilon} \rho \epsilon} \right)^{1/2} \hat{N} \int_0^{\infty} \sqrt{|k_z|} dk_z \rightarrow \infty
 \end{aligned} \tag{6-3-15}$$

where  $\hat{N}$  is the unit vector of  $x - Vt$ , and the fact that :

$$\tau = (z - Vt)/\rho = 1/\gamma_{\epsilon} \tag{6-3-16}$$

on the DWS has been used. Note that  $E$  is real.

The divergent result similar to that of (6-2-14) is again obtained. The reason is that for large  $k_z$ , i.e. large  $\omega$ , the non-dispersive modal is not valid. However, (6-3-15) still gives the direction of  $E$  field and the  $\rho^{-1/2}$  dependence of the field. Note that (6-3-15) only tell the field on the backward Cerenkov cone. This method can not give the  $E$  field for any other point since the far field approximation has been used. However, for this simple case, the field can be calculated also by direct integration of (2-10). The result is ( for  $\beta > 1/\sqrt{\epsilon}$  ) :

$$E(x,t) = -2 \frac{\gamma_{\epsilon} q (x - Vt)}{\epsilon [\gamma_{\epsilon}^2 (z - Vt)^2 - \rho^2]^{3/2}} \tag{6-3-17}$$

which is also divergent at the Cerenkov cone (6-3-16).



## CHAPTER VII

### RADIATION IN AN ISOTROPIC COLD PLASMA

#### 7.1 The far field and the energy flow

The dielectric tensor for an isotropic cold plasma is :

$$\bar{\epsilon}(\mathbf{k}, \omega) = \left[ 1 - \left( \frac{\omega_p}{\omega} \right)^2 \right] \bar{I} \quad (7-1-1)$$

So,

$$\bar{\Delta}(\mathbf{k}, \omega) = (\omega^2 - \omega_p^2 - c^2 k^2) \bar{I} + c^2 \mathbf{k} \mathbf{k} \quad (7-1-2)$$

$$\mathcal{D}(\mathbf{k}, \omega) = (\omega^2 - \omega_p^2)(\omega^2 - \omega_p^2 - c^2 k^2)^2 \quad (7-1-3)$$

$$\bar{\Gamma}(\mathbf{k}, \omega) = (\omega^2 - \omega_p^2 - c^2 k^2) [(\omega^2 - \omega_p^2) \bar{I} - c^2 \mathbf{k} \mathbf{k}] \quad (7-1-4)$$

where  $\omega_p$  is the plasma frequency :

$$\omega_p^2 = 1 - \sum_i 4\pi n_i (Z_i e)^2 / m_i \quad (7-1-5)$$

and  $n_i$ ,  $Z_i e$ ,  $m_i$  are respectively the number density, the charge and the mass of the  $i^{\text{th}}$  species. From (7-1-3) :

$$\frac{\partial \mathcal{D}}{\partial \mathbf{k}} = -4c^2 (\omega^2 - \omega_p^2)(\omega^2 - \omega_p^2 - c^2 k^2) \mathbf{k} \quad (7-1-6)$$

$$\frac{\partial \mathcal{D}}{\partial \omega} = 2\omega (\omega^2 - \omega_p^2 - c^2 k^2)(3\omega^2 - 3\omega_p^2 - c^2 k^2) \quad (7-1-7)$$

and thus obtain the velocity of energy transport :

$$\begin{aligned} \mathbf{V}_E &= - \frac{\partial \mathcal{D}}{\partial \mathbf{k}} / \frac{\partial \mathcal{D}}{\partial \omega} \\ &= 2c^2 (\omega^2 - \omega_p^2) \mathbf{k} / \omega (3\omega^2 - 3\omega_p^2 - c^2 k^2) \end{aligned} \quad (7-1-8)$$

The dispersion relation  $\mathcal{D} = 0$  gives two modes, however, from the above equation, we see that the mode  $\omega^2 = \omega_p^2$  does not propagate, so it will not be considered hereafter. The DWS of the mode  $\omega^2 = \omega_p^2 + c^2 k^2$  is given by (3-2-12). There exists one and only one point of stationary phase for a particular  $(\mathbf{x}, t)$ , as can be seen from the shape of the DWS (Fig. 3-4 ).

The  $\mathbf{V}_E$  of this mode is then, by (7-1-8),

$$\mathbf{V}_E = c^2 \mathbf{k} / \omega$$

The  $\mathbf{k}, \omega$  values and the curvatures at the point of stationary

phase can be calculated in term of  $\tau$  since is of the form of (3-4-1) :

$$k_{\rho} = \omega_o n_o / c (1 + \gamma^2 \tau^2)^{1/2} \quad (7-1-9)$$

$$k_z = \frac{\gamma \omega_o}{c} \left[ \beta - \frac{\gamma n_o \tau}{(1 + \gamma^2 \tau^2)^{1/2}} \right] \quad (7-1-10)$$

$$\omega = \gamma \omega_o \left[ 1 - \frac{\gamma \beta n_o \tau}{(1 + \gamma^2 \tau^2)^{1/2}} \right] \quad (7-1-11)$$

$$k = \frac{\omega_o}{c} \left\{ \gamma^2 \left[ 1 - \frac{\gamma \beta \tau \varepsilon}{(1 + \gamma^2 \tau^2)^{1/2}} \right]^2 - \left( \frac{\omega_p}{\omega_o} \right)^2 \right\}^{1/2} \quad (7-1-12)$$

$$\lambda_{kz} = - \frac{c}{\omega_o n_o \gamma^2} \left[ \frac{1 + \gamma^2 \tau^2}{1 + \tau^2} \right]^{3/2} \quad (7-1-13)$$

$$\lambda_{\phi} = - \frac{c}{\omega_o n_o} \left[ \frac{1 + \gamma^2 \tau^2}{1 + \tau^2} \right]^{1/2} \quad (7-1-14)$$

$$K = \left[ \frac{c}{\omega_o n_o \gamma} \frac{1 + \gamma^2 \tau^2}{1 + \tau^2} \right]^2 \quad (7-1-15)$$

where

$$n_o \equiv [1 - (\omega_p / \omega_o)^2]^{1/2} \quad (7-1-16)$$

Note that if  $1 - (\omega_p / \omega_o)^2 < 0$  the DWS does not exist for any velocity, i.e. there is no radiation field in this case.

We now evaluate the electric far field according to (2-20) using  $J(k)$  being the current density of a linearly oscillating charge in dipole approximation (4-2-45) as an example. Noting that :

$$A = -i \quad (7-1-17)$$

$$\left( \frac{\partial D}{\partial k} + \frac{\partial D}{\partial \omega} v \right)_N = -4c^3 k^2 \omega_o n_o (\omega^2 - \omega_p^2 - c^2 k^2) \left[ \frac{1 + \tau^2}{1 + \gamma^2 \tau^2} \right]^{1/2} \quad (7-1-18)$$

Finally we get :

$$E(x, t) = -\gamma \frac{i q k \times [(k - n k \beta) \times r_o]}{n^2 \rho [1 + \gamma^2 \tau^2]^{1/2}} e^{i(k \cdot x - \omega t)} \quad (7-1-19)$$

$$B(x, t) = n \hat{k} \times E(x, t) \quad (7-1-20)$$



where

$$n \equiv ck/\omega \quad (7-1-21)$$

Note that this expression is very similar to that in vacuum, which can be obtained just by putting  $n = 1$ , i.e.  $\omega_p = 0$ , in (7-1-19).

Lee and Papas (1965) found the radiation of a dipole in a moving isotropic cold plasma with velocity  $V$ . (7-1-19) and (7-1-20) are the same as their result for the  $V = 0$  case. For a general  $V$ , we can compare (7-1-19) and (7-1-20) with the result found <sup>by</sup> moving media method by using Lorentz transformation from the rest frame of the medium to the rest frame of the dipole. However, since Lee and Papas have only considered the case for  $\beta \ll 1$ , so we can not compare their result for a general  $V$ .

The time average energy flow ( Poynting vector ) can be calculated by the formula :

$$\langle S(x,t) \rangle = \frac{c}{16\pi} [ E^* \times B + E \times B^* ] \quad (7-1-22)$$

Using (7-2-19) and (7-2-20) :

$$\begin{aligned} \langle S(x,t) \rangle &= \frac{ck}{8\pi\omega} |E(x,t)|^2 \\ &= \frac{ck}{8\pi\omega} \left( \frac{\gamma q}{n^2 \rho} \right)^2 \frac{|k \times [(k - nk\beta) \times r_o]|^2}{(1 + \gamma^2 \tau^2)} \end{aligned} \quad (7-1-23)$$

$$= \frac{q^2 \omega_o^2 \omega}{8 \pi c^2 \rho^2 (1 + \gamma^2 \tau^2)} [ R^2 - (\hat{k} \cdot R)^2 ] k \quad (7-1-24)$$

where  $R$  has <sup>been</sup> defined by (5-3-3) :

$$R = [ r_o + \frac{\gamma (k \cdot r_o) V}{\omega_o} ] \quad (7-1-25)$$

By (2-29), the radiation power received per unit solid angle subtended at the retarded position is :

$$\frac{dP(x,t)}{d\Omega} = |x - vt'|^2 |\langle S(x,t) \rangle| \quad (7-1-26)$$

and by (2-27) and (2-28) :

$$|x - vt'|^2 = \frac{|x - vt|^2}{|V_E - V|^2} V_E^2 = \frac{\rho^2 (1 + \gamma^2 \tau^2) c^2 k^2}{\omega_o^2 n_o^2} \quad (7-1-27)$$

So,

$$\frac{dP(x,t)}{d\Omega} = \frac{q^2 \omega k [k^2 R^2 - (k \cdot R)^2]}{8 \pi n_o^2} \quad (7-1-28)$$

Note that

$$k \cdot R = (k \cdot r_o) \gamma \omega / \omega_o \quad (7-1-29)$$

(7-1-28) is the received power of the observer. The emitted power per solid angle can be found by (2-30) and (2-31). Now:

$$f = 1 - \frac{V}{V_E} \cos \theta - \frac{V}{V_E^2} \sin \theta \frac{dV_E}{d\theta} \quad (7-1-30)$$

where  $\theta$  is the angle between  $V_E$  and  $V$ , and is the same as the angle between  $k$  and  $V$  in the isotropic case.  $V_E$  is given by (7-1-8). We can calculate it by using

$$\cos \theta = k_z / k, \quad \sin \theta = k_\rho / k \quad (7-1-31)$$

$$\frac{dV_E}{d\theta} = \frac{dV_E}{dk_z} \cdot \frac{dk_z}{d\theta} \quad (7-1-32)$$

$$\frac{dV_E}{dk_z} = \frac{c^2 [\omega (k_\rho \tau + k_z) - k^2 V]}{\omega^2 k} \quad (7-1-33)$$

where  $dk_\rho / dk_z = \tau$  has been used. Also, by (7-1-31) :

$$- \sin \theta \frac{d\theta}{dk_z} = \frac{k_\rho (k_\rho - k_z \tau)}{k^3} \quad (7-1-34)$$

Finally,

$$\frac{dP'}{d\Omega} = f \frac{dP}{d\Omega} = \left( \frac{\omega_o}{c\gamma} \right)^2 \frac{q^2 \omega k [k^2 R^2 - (k \cdot R)^2]}{8 \pi k_\rho (k_\rho - k_z \tau)} \quad (7-1-35)$$

where now  $k_\rho$ ,  $k_z$ ,  $\tau$  should be found as functions of  $\theta$ .

## 7.2 When $\omega_o < \gamma \omega_p$ , some special features

It has been pointed out in § 3.2.4 that when  $\omega_o < \omega_p$ , the whole DWS is above the  $k_z$  (see fig 7-1). Some features can be seen in this case :



### 7.2.1 Complex Doppler effect

For a given  $\theta < \theta_c$ , there exist two wave-vectors  $k_+, k_-$  along the same direction satisfying the Doppler equation :

$$\omega_{\pm} = \frac{\omega_o}{\gamma} + k_{\pm}(\omega) V \cos \theta \quad (7-2-1)$$

This is called the complex Doppler effect which was first found by Frank (1943).

Note that this effect can not happen if  $k(\omega)$  is a linear function of  $\omega$ , i.e. non-dispersive media.

By (7-2-1) and the dispersion relation :

$$n^2 = \frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2} \quad (7-2-2)$$

$k$ ,  $\omega$  and  $n$  can be found as functions of  $\theta$  :

$$ck = \frac{\frac{\omega_o}{\gamma} \beta \cos \theta \pm \sqrt{\omega_o^2 (n_o^2 - \beta^2) + \omega_p^2 \beta^2 \cos^2 \theta}}{1 - \beta^2 \cos^2 \theta} \quad (7-2-3)$$

$$\omega = \frac{\frac{\omega_o}{\gamma} \pm \beta \cos \theta \sqrt{\omega_o^2 (n_o^2 - \beta^2) + \omega_p^2 \beta^2 \cos^2 \theta}}{1 - \beta^2 \cos^2 \theta} \quad (7-2-4)$$

$$n = \frac{\omega_p^2 \beta \cos \theta \pm \frac{\omega_o}{\gamma} \sqrt{\omega_o^2 (n_o^2 - \beta^2) + \omega_p^2 \beta^2 \cos^2 \theta}}{(\omega_o / \gamma)^2 + \omega_p^2 \beta^2 \cos^2 \theta} \quad (7-2-5)$$

Note that  $V_E = cn$ . Graphs for  $k$  vs  $\theta$  (DWS) and  $V_E$  vs  $\theta$  are plotted in Figs. 7-1.

So, we see that when

$$\omega_o^2 (n_o^2 - \beta^2) + \omega_p^2 \beta^2 \cos^2 \theta < 0 \quad (7-2-6)$$

$$\text{i.e. } \cos^2 \theta < \cos^2 \theta_c \equiv \frac{\omega_o^2 (\beta^2 - n_o^2)}{\omega_p^2 \beta^2} \quad (7-2-7)$$

no  $(k, \omega)$  exists. Note also that this effect only occurs when  $\beta > n_o$ .

Although the two wave-vectors  $k_+$  and  $k_-$  are in the same direction, the two waves are not observed at the same time because the group velocity of the two waves is not the same :

$$V_{E\pm} = c n_{\pm} \hat{k} \quad (7-2-8)$$

$k_+$  propagates faster than  $k_-$  since  $n_+ > n_-$ . So, a radiating source moving at constant velocity  $V \hat{z}$  in the isotropic cold plasma will be seen by an observer at a fixed position in the plasma that it is first coming from the  $-\hat{z}$  direction and then "turn back" to the same direction at some instant.

Note that since  $n_o = [1 - (\omega_p/\omega_o)^2]^{1/2}$  can be very small if  $\omega_o \cong \omega_p$ , this effect can be observed even for low velocity source.

### 7.2.2 All-forward radiation

It can be seen easily that all the radiation is confined within a cone of angle  $\theta_c$  and all forward. By (7-2-6),

$$\tan \theta_c = n_o / \gamma (\beta^2 - n_o^2)^{1/2} \quad (7-2-9)$$

for  $\beta \rightarrow 1$ ,

$$\tan \theta_c \cong \theta_c \cong n_o / \gamma (1 - n_o^2)^{1/2} = n_o \omega_o / \gamma \omega_p \rightarrow 0 \quad (7-2-10)$$

so the solid angle of the cone that the radiation exist becomes :

$$\Delta \Omega \cong \pi (n_o \omega_o / \gamma \omega_p)^2 \rightarrow 0 \quad (7-2-11)$$

However, the Poynting vector when  $\theta = 0$  becomes infinite as can be seen by (7-1-24) and (7-1-25) :

$$\langle S \rangle_{\theta=0} \rightarrow \frac{q^2 \rho_o^2 \omega_o^4 \gamma^2 (1 \pm n_o)^2 \hat{z}}{8 \pi c^3 (z - Vt)^2} \rightarrow \infty \quad (7-2-12)$$

where

$$\rho_o = (x_o^2 + y_o^2)^{1/2} \quad (7-2-13)$$

So, the total power :

$$\begin{aligned} W &\cong \hat{z} \cdot \langle S \rangle_{\theta=0} (z - Vt)^2 \Delta \Omega \\ &\cong \frac{q^2 \rho_o^2 \omega_o^6 n_o^2 (1 \pm n_o)^2}{8 c^3 \omega_p^2} \end{aligned} \quad (7-2-14)$$

which is finite. We see that at relativistic case, there is



an intensive wave beam along the z-axis radiated from the source.

### 7.2.3 Anomalous Doppler effect

In the ordinary Doppler<sup>effect</sup>,  $\omega$  decreases as  $\theta$  increase. However, for the  $\omega_-$  branch of the complex Doppler effect,  $\omega$  increase as  $\theta$  increase. This effect has been termed as the anomalous Doppler effect (Ginzburg, 1959). Note that this effect also occurs in an isotropic non-dispersive medium when  $\beta > 1 / \sqrt{\epsilon}$  (see  $k_-$  of Fig. 6-2).

### 7-2-4 Red-shift of forward radiation

In the ordinary Doppler effect, the forward radiation of the moving source must be blue-shifted, e.g. in the vacuum case, we have ,

$$\omega = \omega_0 \left( \frac{1 + \beta}{1 - \beta} \right)^{1/2} \quad (7-2-15)$$

However, by (7-2-4), the frequency of the forward radiation ( $\theta = 0$ ) are :

$$\omega = \gamma \omega_0 (1 \pm \beta n_0) \quad (7-2-16)$$

We see that  $\omega_+$  is always blue-shifted, but  $\omega_-$  may be red-shifted since  $\gamma(1 - \beta n_0)$  can be smaller than 1, i.e.

$$n_0 < \beta < 2n_0 / (1 + n_0^2) \quad (7-2-17)$$

Note that the group velocity of this red-shifted radiation is always smaller than the velocity of the source since it is observer for  $\tau = \infty$ , i.e. after the source has past through the observer if the source is moving at a constant velocity.

### 7.2.5 Infinite emitted power per unit solid angle at $\theta_c$

From (7-1-44), we see that the emitted power per unit solid angle  $dP / d\Omega$  becomes infinite when

$$\frac{k}{k_z} \frac{\rho}{\rho_z} = \tau = \frac{dk}{dk_z} \frac{\rho}{\rho_z} \quad (7-2-18)$$

which is just the condition for  $\theta = \theta_c$  as can be seen easily from Fig. 7-1 . The physical reason for this infinity is that at  $\theta = \theta_c$ , a very small solid angle  $d\Omega$  contains a very large area of the DWS, so the radiation emitted from this solid angle  $d\Omega$  is very large compared with that of the other  $\theta < \theta_c$ . On the other hand, the received power per unit solid angle is not infinite at  $\theta = \theta_c$ . This is because the group velocity of those waves inside the solid at  $\theta = \theta_c$  is different, and so they are received by different time at a fixed observation point.

The total emitted power is also finite although  $dP'/d\Omega$  is infinite at  $\theta = \theta_c$ . The total emitted power is :

$$W' = \int \frac{dP'}{d\Omega} \sin\theta \, d\theta \, d\phi \quad (7-2-17)$$

for  $\theta \cong \theta_c$ ,

$$k_\rho - k_z \tau \cong k_z (\tan\theta - \tau) \cong k_z \left. \frac{d\tau}{d\theta} \right|_{\theta=\theta_c} (\theta - \theta_c) \quad (7-2-18)$$

$$\text{and } \left. \frac{d\tau}{d\theta} \right|_{\theta=\theta_c} = \frac{d}{d\theta} (k_\rho/k_z) = 1 / \cos^2 \theta_c \quad (7-2-19)$$

then, by (7-1-35) :

$$\left. \frac{dP'}{d\Omega} \sin\theta \, d\theta \right|_{\theta=\theta_c} \cong \left( \frac{\omega_0}{c\gamma} \right)^2 \frac{\omega k [k^2 p^2 - (k \cdot p)^2]}{8\pi k_z k_\rho (1 / \cos^2 \theta_c)} \quad (7-2-20)$$

is finite. So,  $W'$  is not infinite.

#### Final remarks

The above features may also appear for a moving radiating source in other dispersive media. Actually, the first paper (as far as we know) discussing the complex Doppler effect used the medium  $n(\omega) = n_o^2 + \omega_p^2/(\omega_o^2 - \omega^2)$  in the discussion (Frank, 1943). However, isotropic cold plasma may give the simplest and clearest illustration of these features, especially with the help of the DWS.



## CHAPTER VIII

Non-Dispersive

### RADIATION IN ISOTROPIC UNIAXIAL MEDIA

#### 8.1 Dispersion relation

The foregoing consideration is confined to isotropic media, we now extend to the anisotropic case. The simplest case is an non-dispersive uniaxial medium with dielectric tensor :

$$\bar{\epsilon} = \epsilon_{\perp} \bar{I} + (\epsilon_{\parallel} - \epsilon_{\perp}) \hat{z}\hat{z} \quad (8-1-1)$$

So,

$$\bar{\Delta} = (\epsilon_{\perp} \omega^2 - c^2 k^2) \bar{I} + (\epsilon_{\parallel} - \epsilon_{\perp}) \omega^2 \hat{z}\hat{z} + c^2 k k \quad (8-1-2)$$

$$\mathcal{D} = \omega^2 (\epsilon_{\perp} \omega^2 - c^2 k^2) (\omega^2 \epsilon_{\parallel} \epsilon_{\perp} - \epsilon_{\perp} c^2 k_{\rho}^2 - \epsilon_{\parallel} c^2 k_z^2) \quad (8-1-3)$$

$$\begin{aligned} \bar{\Gamma} = & (\epsilon_{\perp} \omega^2 - c^2 k^2) [\epsilon_{\parallel} \omega^2 \bar{I} + (\epsilon_{\perp} - \epsilon_{\parallel}) \omega^2 \hat{z}\hat{z} - c^2 k k] \\ & + c^2 k_{\rho}^2 \omega^2 (\epsilon_{\parallel} - \epsilon_{\perp}) \hat{\phi}\hat{\phi} \end{aligned} \quad (8-1-4)$$

Note that

$$\hat{\phi} = -\sin\phi \hat{x} + \cos\phi \hat{y} \quad (8-1-5)$$

$$\hat{\rho} = \cos\phi \hat{x} + \sin\phi \hat{y} \quad (8-1-6)$$

From (8-1-3), we see that there exist two modes :

$$\text{ordinary mode (o): } \epsilon_{\perp} \omega^2 - c^2 k^2 = 0 \quad (8-1-7)$$

$$\text{extraordinary mode (e): } \omega^2 \epsilon_{\parallel} \epsilon_{\perp} - \epsilon_{\perp} c^2 k_{\rho}^2 - \epsilon_{\parallel} c^2 k_z^2 = 0 \quad (8-1-8)$$

For the (o) mode, we have :

$$\frac{\partial \mathcal{D}}{\partial k} = -2c^4 k_{\rho}^2 (\epsilon_{\parallel} - \epsilon_{\perp}) \omega^2 k \quad (8-1-9)$$

$$\frac{\partial \mathcal{D}}{\partial \omega} = 2c^2 k_{\rho}^2 \epsilon_{\perp} (\epsilon_{\parallel} - \epsilon_{\perp}) \omega^3 \quad (8-1-10)$$

and, for the (e) mode ,

$$\frac{\partial \mathcal{D}}{\partial k} = 2c^4 k_{\rho}^2 (\epsilon_{\parallel} - \epsilon_{\perp}) \omega^2 \left( \frac{1}{\ell} k_{\rho} \hat{\rho} + k_z \hat{z} \right) \quad (8-1-11)$$

$$\frac{\partial \mathcal{D}}{\partial \omega} = -2c^2 k_{\rho}^2 \epsilon_{\perp} (\epsilon_{\parallel} - \epsilon_{\perp}) \omega^3 \quad (8-1-12)$$

where

$$\ell \equiv \epsilon_{\parallel} / \epsilon_{\perp} \quad (8-1-13)$$

Note that  $\epsilon_{\parallel} \neq \epsilon_{\perp}$  is assumed. For  $\epsilon_{\parallel} = \epsilon_{\perp}$ , the medium reduces to the isotropic one which has already been discussed in chapter VI.

The DWS of the two modes for  $V$  along  $\hat{z}$  are given by (3-2-9) and (3-2-10) which are of the same form as that for an isotropic medium except that when  $\beta = 0$ , the DWS of the (e) mode is still an ellipsoid but not a sphere :

$$c^2 k_{\rho}^2 = \epsilon_{\parallel} \omega_o^2 - \ell c^2 k_z^2 \quad (\beta = 0) \quad (8-1-14)$$

So, it is an anisotropic mode. This can also be seen from the group velocity :

$$(o) : V_E = c k / \sqrt{\epsilon_{\perp}} \quad (8-1-15)$$

$$(e) : V_E = c^2 (\epsilon_{\perp} k_{\rho} \hat{\rho} + \epsilon_{\parallel} k_z \hat{z}) / \omega \epsilon_{\parallel} \epsilon_{\perp} \quad (8-1-16)$$

We can also rewrite (8-1-4) for different modes :

$$(o) : \bar{\Gamma} = c^2 k_{\rho}^2 \omega^2 (\epsilon_{\parallel} - \epsilon_{\perp}) \hat{\phi} \hat{\phi} \quad (8-1-17)$$

$$(e) : \bar{\Gamma} = -c^2 k_{\rho}^2 \frac{1}{\epsilon_{\parallel}} (\epsilon_{\parallel} - \epsilon_{\perp}) [\omega^2 (\epsilon_{\parallel} \hat{\rho} \hat{\rho} + \epsilon_{\perp} \hat{z} \hat{z}) - c^2 k k] \quad (8-1-18)$$

So, we see that for the (o) mode,  $\bar{\Gamma} \cdot J$  just picks up the  $\phi$  component of the current density and the field is also in the  $\hat{\phi}$  direction ; while for the (e) mode, it picks up the  $\rho$  and  $z$  component of  $J$  and  $E$  field is also lying on the plane determined by these two directions. Thus the electric fields of the two modes are perpendicular to each other. Since ,

$$B_e = n_o \hat{k}_e \times E_e = B_e \hat{\phi} \quad (8-1-19)$$

So,

$$E_o \times B_e = 0 \quad (8-1-20)$$

Also,

$$B_o = n_o \hat{k}_o \times E_o = B_o \hat{\phi} \perp \hat{\phi} \quad (8-1-21)$$

Thus,

$$E_e \times B_o \parallel \hat{\phi} \quad (8-1-22)$$



This shows that the interference of the two modes does not carry energy away from the source in general. Actually, Lai and Chan have proved that  $\mathbf{E}_0 \times \mathbf{B}_0 = 0$  for the  $\beta = 0$  case. This can be seen by noting that  $\mathbf{V}_{E_0} // \mathbf{V}_{E_0} // \mathbf{x}$  in this case and thus  $\mathbf{E}_0 // \mathbf{B}_0$ . For  $\beta \neq 0$ , the two waves of the two modes received by an observer is in general having different frequency except for some special cases (see Fig. 8-1) , so the mixed mode interference effect is not important here , and thus we can treat the radiation power separately for each mode.

## 8.2 V along symmetric axis with $V < c / \sqrt{\epsilon_{\perp}}$

Consider a radiating source moving along the symmetric axis  $\hat{z}$  with constant velocity  $V$ . When  $\beta < 1/\sqrt{\epsilon_{\perp}}$  , the DWS of both modes are ellipsoidal (§ 3.2.3). The values of  $k, \omega$  and  $K$  at the point of stationary phase point for the (o) mode are given by (6-1-5) - (6-1-8) and (6-1-10) - (6-1-12) with  $\epsilon \rightarrow \epsilon_{\perp}$ . For the (o) mode :

$$k_{\rho} = \gamma_{\epsilon} \omega_0 / \gamma c [\ell \epsilon_{\parallel} / (\ell + \gamma_{\epsilon}^2 \tau^2)]^{1/2} \quad (8-2-1)$$

$$k_z = \frac{\gamma_{\epsilon}^2 \omega_0}{\gamma c} \left[ \epsilon_{\perp} \beta - \frac{\gamma_{\epsilon} \tau \sqrt{\epsilon_{\perp}}}{[\ell + \gamma_{\epsilon}^2 \tau^2]^{1/2}} \right] \quad (8-2-2)$$

$$\omega = \frac{\gamma_{\epsilon}^2 \omega_0}{\gamma} \left[ 1 - \frac{\gamma_{\epsilon} \beta \tau \sqrt{\epsilon_{\perp}}}{[\ell + \gamma_{\epsilon}^2 \tau^2]^{1/2}} \right] \quad (8-2-3)$$

$$\lambda_{kz} = - \frac{c \gamma}{\omega_0 \gamma_{\epsilon}^3 \sqrt{\ell \epsilon_{\parallel}}} \left[ \frac{\ell + \gamma_{\epsilon}^2 \tau^2}{1 + \tau^2} \right]^{3/2} \quad (8-2-4)$$

$$\lambda_{\phi} = - \frac{c \gamma}{\omega_0 \gamma_{\epsilon} \sqrt{\ell \epsilon_{\parallel}}} \left[ \frac{\ell + \gamma_{\epsilon}^2 \tau^2}{1 + \tau^2} \right]^{1/2} \quad (8-2-5)$$

$$K = \frac{1}{\ell \epsilon_{\parallel}} \left[ \frac{c \gamma}{\omega_0 \gamma_{\epsilon}^2} \frac{\ell + \gamma_{\epsilon}^2 \tau^2}{1 + \tau^2} \right]^2 \quad (8-2-6)$$

where now,

$$\gamma_{\varepsilon}^2 \equiv 1 / |1 - \varepsilon_{\perp} \beta^2| \quad (8-2-7)$$

Note that these quantities reduce to those in isotropic case if we put  $\ell = 1$ . Similar to (6-1-15), we have

$$\left( \frac{\partial \mathcal{D}}{\partial k} + \frac{\partial \mathcal{D}}{\partial \omega} \mathbf{V} \right)_{N_0} = -2c^4 k_{\rho}^3 (\varepsilon_{\parallel} - \varepsilon_{\perp}) \omega^2 (1 + \tau^2) \quad (8-2-9)$$

$$\left( \frac{\partial \mathcal{D}}{\partial k} + \frac{\partial \mathcal{D}}{\partial \omega} \mathbf{V} \right)_{N_0} = 2c^4 k_{\rho}^3 (\varepsilon_{\parallel} - \varepsilon_{\perp}) \omega^2 (1 + \tau^2) / \ell \quad (8-2-10)$$

The electric far field of both modes are then found :

$$\mathbf{E}_0(\mathbf{x}, t) = \frac{2i\gamma_{\varepsilon} \omega \hat{\phi} \cdot \hat{\phi} \cdot \mathbf{J}(\mathbf{k})}{c^2 \rho (1 + \gamma_{\varepsilon}^2 \tau^2)^{1/2}} e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} \quad (8-2-11)$$

$$\begin{aligned} \mathbf{E}_e &= \frac{2i\gamma_{\varepsilon} [\omega^2 (\varepsilon_{\parallel} \hat{\rho} \hat{\rho} + \varepsilon_{\perp} \hat{z} \hat{z}) - c^2 \mathbf{k} \mathbf{k}] \cdot \mathbf{J}(\mathbf{k})}{\varepsilon_{\perp} \omega \rho (\ell + \gamma_{\varepsilon}^2 \tau^2)^{1/2}} e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} \\ &= \frac{2i\gamma_{\varepsilon} [k_z (\ell k_z J_{\rho} - k_{\rho} J_z) - k_{\rho} (\frac{1}{\ell} k_{\rho} J_z - k_z J_{\rho}) \hat{z}]}{\varepsilon_{\perp} \omega \rho (\ell + \gamma_{\varepsilon}^2 \tau^2)^{1/2}} e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} \end{aligned} \quad (8-2-12)$$

Note that  $\mathbf{k}$ ,  $\omega$  in (8-2-11) and (8-2-12) are different functions of  $\tau$ .

We see that although the dispersion relation of the ordinary mode is the same as that of the isotropic medium, the radiation field is quite different in these two cases as can be seen by comparing (8-2-12) to (6-1-18). If we now let  $\ell = 1$ , i.e.  $\varepsilon_{\parallel} = \varepsilon_{\perp} = \varepsilon$ , the total field is

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_0 + \mathbf{E}_e \\ &= \frac{2i\gamma_{\varepsilon} q(k^2 \bar{\mathbf{I}} - \mathbf{k} \mathbf{k}) \cdot \mathbf{J}(\mathbf{k})}{\varepsilon \omega \rho [1 + \gamma_{\varepsilon}^2 \tau^2]^{1/2}} e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} \end{aligned} \quad (8-2-13)$$

which is just the field found in the isotropic case, i.e. (6-1-18), where we now use a general  $\mathbf{J}(\mathbf{k})$ . This gives a check to the result. We also note that the fields of the two modes do not reduce to that of isotropic case separately in this limit, although their dispersion relation both tend to that of the same isotropic mode. Instead, the total field of the two modes gives the correct limiting result.



Another check can be done by putting  $\beta = 1$  to compare with the well known result (e.g. Chen, 1983, eqn. 10.82). They are really the same if only the far field ( $E \propto 1/r$ ) is considered.

As has been discussed in § 8.1, the interference between the two modes is not important, so the time-averaged Poynting vector can be found by each mode separately :

$$\langle S \rangle_o = \frac{\gamma_\epsilon^2 |J_\phi|^2 \omega k}{2\pi c^3 \rho^2 [1 + \gamma_\epsilon^2 \tau^2]} \quad (8-2-14)$$

$$\langle S \rangle_e = \frac{\gamma_\epsilon^2 |\epsilon_{\parallel} k_z J_\rho - \epsilon_{\perp} k_\rho J_z|^2 [\epsilon_{\perp} k_\rho \hat{\rho} + \epsilon_{\parallel} k_z \hat{z}]}{2\pi \epsilon_{\perp}^3 \epsilon_{\parallel} \omega \rho^2 [1 + \gamma_\epsilon^2 \tau^2]} \quad (8-2-15)$$

If we put  $\ell = 1$ , then the total Poynting vector reduces to the result of isotropic case :

$$\langle S \rangle = \langle S \rangle_e + \langle S \rangle_o = \frac{\gamma_\epsilon^2 [k^2 J^2 - (k \cdot J)^2] \hat{k}}{2\pi \sqrt{\epsilon} c \rho^2 [1 + \gamma_\epsilon^2 \tau^2]} \quad (8-2-16)$$

The power received per unit solid angle are :

$$\left( \frac{dP}{d\Omega} \right)_o = \frac{\gamma_\epsilon^2 |J_\phi|^2 \omega^3 k}{2\pi c^2 \omega_o^2} \quad (8-2-17)$$

$$\left( \frac{dP}{d\Omega} \right)_e = \frac{\gamma_\epsilon^2 c^2 |\epsilon_{\parallel} k_z J_\rho - \epsilon_{\perp} k_\rho J_z|^2 [\epsilon_{\perp}^2 k_\rho^2 + \epsilon_{\parallel}^2 k_z^2]^{3/2}}{2\pi \epsilon_{\perp}^4 \epsilon_{\parallel}^3 \omega \omega_o^2 \ell} \quad (8-2-18)$$

The power emitted per unit solid angle for the (o) mode can be found easily since the  $f$  factor is just the same as that of the isotropic case, so

$$\left( \frac{dP'}{d\Omega} \right)_o = \frac{|J_\phi|^2 \omega_o^2}{\pi c^3 \sqrt{\epsilon_{\perp}} \gamma^2 (1 - \beta \sqrt{\epsilon_{\perp}} \cos \theta)^3} \quad (8-2-19)$$

For the (e) mode, the  $f$  factor is found by (2-31):

$$f = \frac{(k_z \tau - k_\rho)(1 - \epsilon_{\perp} \beta^2) - (\epsilon_{\perp} \omega_o \beta \tau / \gamma c)}{k_z \tau - k_\rho} \quad (8-2-20)$$

where  $k_z$ ,  $k_\rho$  and  $\tau$  are expressed as functions of  $\theta$  (angle between  $V$  and  $V_E$ ) :

$$k_z = \frac{\omega_o}{\gamma c} \left[ \frac{-\varepsilon_{\perp} \beta \pm [\varepsilon_{\perp} + \varepsilon_{\parallel} \tan^2 \theta]^{1/2}}{\ell \tan^2 \theta + 1 - \beta^2 \varepsilon_{\perp}} \right] \quad (8-2-21)$$

$$k_{\rho} = \ell k_z \tan \theta \quad (8-2-22)$$

$$\tau = \frac{\omega_o \varepsilon_{\parallel} \beta}{\gamma c k_{\rho}} + \frac{\beta^2 \varepsilon_{\perp} - 1}{\tan \theta} \quad (8-2-23)$$

So, the emitted power per unit solid angle for the (e) mode is :

$$\left( \frac{dP'}{d\Omega} \right)_e = f \left( \frac{dP}{d\Omega} \right)_e$$

with  $f$  given by (8-2-20) and note that  $\omega = \frac{\omega_o}{\gamma} + k_z V$  is also expressed in  $\theta$ .

## 8.2 V along symmetric axis with $V > c / \sqrt{\varepsilon_{\perp}}$

When  $\beta > 1/\sqrt{\varepsilon_{\perp}}$ , DWS of both modes become hyperboloid. The values of  $k, \omega, K$  at the stationary phase points for the (o) mode are given by (6-2-1) - (6-2-5) with  $\varepsilon = \varepsilon_{\perp}$ . For the (e) mode :

$$k_{\rho} = \gamma_{\varepsilon} \omega_o / \gamma c [\varepsilon_{\parallel} \ell (-\ell + \gamma_{\varepsilon}^2 \tau^2)]^{1/2} \quad (8-3-1)$$

$$k_z = \frac{\gamma_{\varepsilon}^2 \omega_o}{\gamma c} \left[ -\varepsilon_{\perp} \beta + \frac{\sigma \gamma_{\varepsilon} \tau \sqrt{\varepsilon_{\perp}}}{[-\ell + \gamma_{\varepsilon}^2 \tau^2]^{1/2}} \right] \quad (8-3-2)$$

$$\omega = \frac{\gamma_{\varepsilon}^2 \omega_o}{\gamma} \left[ -1 + \frac{\sigma \gamma_{\varepsilon} \beta \tau \sqrt{\varepsilon_{\perp}}}{[-\ell + \gamma_{\varepsilon}^2 \tau^2]^{1/2}} \right] \quad (8-3-3)$$

$$\lambda_{kz} = -\sigma \frac{c \gamma}{\omega_o \gamma_{\varepsilon}^3 \sqrt{\ell \varepsilon_{\parallel}}} \left[ \frac{-\ell + \gamma_{\varepsilon}^2 \tau^2}{1 + \tau^2} \right]^{3/2} \quad (8-3-4)$$

$$\lambda_{\phi} = -\sigma \frac{c \gamma}{\omega_o \gamma_{\varepsilon} \sqrt{\ell \varepsilon_{\parallel}}} \left[ \frac{-\ell + \gamma_{\varepsilon}^2 \tau^2}{1 + \tau^2} \right]^{1/2} \quad (8-3-5)$$

$$K = \frac{1}{\ell \varepsilon_{\parallel}} \left[ \frac{c \gamma}{\omega_o \gamma_{\varepsilon}^2} \frac{-\ell + \gamma_{\varepsilon}^2 \tau^2}{1 + \tau^2} \right]^2 \quad (8-3-6)$$



Note that  $\sigma = 1, -1$  for the upper branch or the lower branch and  $\mathbf{k} = \sigma k_\rho \hat{\rho} + k_z \hat{z}$ . By (8-3-4) and (8-3-5) :

$$A^\sigma = -i\sigma \quad (8-3-7)$$

(8-2-9) and (8-2-10) can still be used but a factor  $\sigma$  should be multiplied to them. Finally, we found that :

$$E_o = \sum_{\sigma} \frac{2i\gamma_\varepsilon \omega_\sigma \hat{\phi} \cdot \hat{\phi} J(k)}{c^2 \rho (-1 + \gamma_\varepsilon^2 \tau^2)^{1/2}} e^{i(\mathbf{k}_\sigma \cdot \mathbf{x} - \omega_\sigma t)} \quad (8-3-8)$$

$$E_e = \sum_{\sigma} \frac{2i\gamma_\varepsilon [\omega_\sigma^2 (\varepsilon_\parallel \hat{\rho}\hat{\rho} + \varepsilon_\perp \hat{z}\hat{z}) - c^2 \mathbf{k}_\sigma \mathbf{k}_\sigma] \cdot \mathbf{J}(\mathbf{k}_\sigma)}{\varepsilon_\perp \omega_\sigma \rho (-1 + \gamma_\varepsilon^2 \tau^2)^{1/2}} e^{i(\mathbf{k}_\sigma \cdot \mathbf{x} - \omega_\sigma t)} \quad (8-3-9)$$

Similar to the discussion before (8-2-13), if we let  $\ell = 1$ , then the summation of the field of the two modes gives the result of the isotropic medium, i.e. (6-2-14) :

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_o + \mathbf{E}_e \\ &= \sum_{\sigma} \frac{2i\gamma_\varepsilon \rho (k_\sigma^2 \bar{\mathbf{I}} - \mathbf{k}_\sigma \mathbf{k}_\sigma) \cdot \mathbf{J}(\mathbf{k}_\sigma)}{\varepsilon \omega_\sigma \rho [-1 + \gamma_\varepsilon^2 \tau^2]^{1/2}} e^{i(\mathbf{k}_\sigma \cdot \mathbf{x} - \omega_\sigma t)} \end{aligned} \quad (8-3-10)$$

Note that the fields given by (8-3-8) and (8-3-9) exist only inside the backward Cerenkov cones :

$$(o) : \tau > 1 / \gamma_\varepsilon \quad (8-3-11)$$

$$(e) : \tau > \sqrt{\ell} / \gamma_\varepsilon \quad (8-3-12)$$

respectively.

The Cerenkov cone for the (o) mode (8-3-11) is similar to that discussed in § 6-2. On the Cerenkov cone surface of the (e) mode (8-3-12), see Fig. 8-2 ,

$$\tau = \tan \psi = \frac{\sqrt{\ell}}{\gamma_\varepsilon} = \frac{k_\rho}{k_z} \quad (8-3-13)$$

where  $\psi$  is the angle between  $\mathbf{k}$  and  $\mathbf{V}$ . Note that on this surface, by (8-3-1) - (8-3-4) :

$$k_z \rightarrow \frac{\gamma_\varepsilon^3 \omega_\sigma \sigma \tau \sqrt{\varepsilon_\perp}}{\gamma c [-1 + \gamma_\varepsilon^2 \tau^2]^{1/2}} \quad (8-3-14)$$

$$\omega \rightarrow \frac{\gamma_{\epsilon}^3 \omega_0 \sigma \beta \tau \sqrt{\epsilon}_{\perp}}{\gamma [-\ell + \gamma_{\epsilon}^2 \tau^2]^{1/2}} \quad (8-3-15)$$

$$\frac{\omega}{k} \rightarrow c \beta \gamma_{\epsilon} / (\gamma_{\epsilon}^2 + \ell)^{1/2} \quad (8-3-16)$$

Thus, by (8-3-13) :

$$\begin{aligned} \cos \psi &= \gamma_{\epsilon} / (\gamma_{\epsilon}^2 + \ell)^{1/2} = \frac{\omega}{k} / c\beta \\ &= \frac{\text{phase velocity of the radiation}}{\text{source's velocity}} \end{aligned} \quad (8-3-17)$$

This is just the same relation as that of the isotropic case, i.e. (6-2-7), although now the (e) mode is anisotropic and the phase velocity varies as  $\psi$ .

The time-averaged Poynting vector and radiation power can be found by similar method as the above section for each mode and each  $\alpha$  separately since the interference between the two  $\alpha$ 's is not important.

Discussion on Cerenkov radiation can be made similar to that of § 6.3, but it will not be given here. From the above discussion we know that the condition for the occurrence of Cerenkov radiation is the same in the two modes. However the angle of the Cerenkov cone is different. Moreover, if  $J(k) = q V \hat{z}$ , then from (8-1-21), there is no Cerenkov radiation for the (o) mode.

#### 8.4 $V$ not parallel to the symmetric axis

In the above section, we assumed that the velocity of the source is parallel to the axis of symmetric axis of the medium so that the DWS is still cylindrically symmetric about the axis. In this section, we will see what happens for a general  $V$ . Consider

$$V = V_z \hat{z} + V_x \hat{x} \quad (8-4-1)$$

This  $V$  is general since we have the freedom to choose the



#### 8.4 V not parallel to the symmetric axis

In the above sections, we assumed that the velocity of the source is parallel to the symmetric axis of the medium so that the DWS is still cylindrically symmetric about the axis. In this section, we will see what happens for a general V. Consider

$$\mathbf{V} = V_z \hat{z} + V_x \hat{x} \quad (8-4-1)$$

This V is general since we have the freedom to choose the  $\hat{x}$ -axis.

For the (o) mode, since it is isotropic, DWS is still cylindrically symmetric so that the radiation field found in the above sections, i.e. (8-2-11) and (8-3-8), are still valid.

However, for the (e) mode, DWS is no longer given by (3-2-10) and so  $k, \omega$  found in the above sections can not be used in this case. Instead, the DWS is changed to :

$$\left(\frac{\omega}{\gamma} + k_z V_z + k_x V_x\right)^2 \varepsilon_{\perp} \varepsilon_{\parallel} = \varepsilon_{\perp} c^2 k_{\rho}^2 + \varepsilon_{\parallel} c^2 k_z^2 \quad (8-4-2)$$

or it can be written as :

$$k_y^2 = \left(\frac{\omega}{\gamma} + k_z \beta_z + k_x \beta_x\right)^2 \varepsilon_{\parallel} - k_x^2 + k_z^2 \quad (8-4-3)$$

which is still a quadratic surface, but is not cylindrically symmetric and is in the form of :

$$k_y^2 = A k_z^2 + B k_x^2 + C k_z k_x + D k_z + E k_x + F \quad (8-4-4)$$

where A, B, C, D, E, F are constants. So,

$$\mathbf{k} = k_x \hat{x} + k_y(k_x, k_z) \hat{y} + k_z \hat{z} \quad (8-4-5)$$

We can use  $k_z, k_x$  to be the coordinate variables. However, they are not orthogonal any more in general. This does not mean that (2-20) can not be used now, since the Gaussian curvature K is independent of the choice of coordinate. Only

the result obtained in § 3.3 can not be used in this case, instead, we must do it again in a more general manner. By (8-4-5):

$$\frac{\partial k}{\partial k_z} = k'_y \hat{y} + \hat{z} \quad (8-4-6)$$

where now ' means  $\partial/\partial k_z$ .

$$\frac{\partial k}{\partial k_x} = \hat{x} + k'_y \hat{y} \quad (8-4-7)$$

where ' means  $\partial/\partial k_x$ .

So,

$$\frac{\partial k}{\partial k_z} \cdot \frac{\partial k}{\partial k_x} = k'_y \dot{k}_y \quad (8-4-8)$$

which is non-zero in general.

$$\begin{aligned} \hat{N} &= \pm \frac{\frac{\partial k}{\partial k_z} \times \frac{\partial k}{\partial k_x}}{\left| \frac{\partial k}{\partial k_z} \times \frac{\partial k}{\partial k_x} \right|} \\ &= \pm \frac{-\dot{k}_y \hat{x} + \hat{y} - k'_y \hat{z}}{\sqrt{1 + \dot{k}_y^2 + k'^2_y}} \end{aligned} \quad (8-4-9)$$

Note that  $\hat{N}$  is actually defined as the unit vector of  $V_E - V$ , which is normal to the DWS. The magnitude of the Gaussian curvature can then be calculated by the following formula :

$$K = \left| \frac{\left( \frac{\partial^2 k}{\partial k_z^2} \cdot \hat{N} \right) \left( \frac{\partial^2 k}{\partial k_x^2} \cdot \hat{N} \right) - \left( \frac{\partial^2 k}{\partial k_z \partial k_x} \cdot \hat{N} \right)^2}{\left( \frac{\partial k}{\partial k_x} \times \frac{\partial k}{\partial k_z} \right)^2} \right| \quad (8-4-10)$$

Using

$$\partial^2 k / \partial k_z^2 = k''_y \hat{y} \quad (8-4-11)$$

$$\partial^2 k / \partial k_z \partial k_x = \dot{k}'_y \hat{y} \quad (8-4-12)$$

$$\partial^2 k / \partial k_x^2 = \dot{k}_y \hat{y} \quad (8-4-13)$$

So,



$$K = \left| \frac{k_y'' k_y'' - k_y'^2}{1 + k_y'^2 + k_y'^2} \right| \quad (8-4-14)$$

Note that if we now let the surface be cylindricallly symmetric, (8-4-16) gives the same result as (3-3-10) as expected.

Note that  $\hat{N}$  at the point of stationary phase is parallel to  $x - Vt$ . Let

$$x = x\hat{x} + y\hat{y} + z\hat{z} \quad (8-4-15)$$

$$\frac{x-Vt}{|x-Vt|} = \frac{-\tau_x \hat{x} + \hat{y} - \tau_z \hat{z}}{\sqrt{1 + \tau_x^2 + \tau_z^2}} \quad (8-4-16)$$

where

$$\tau_x \equiv (Vt - x)/y \quad (8-4-17)$$

$$\tau_z \equiv (Vt - z)/y \quad (8-4-18)$$

by comparing with (8-4-11) , we found :

$$k_y' = \pm \tau_z \quad (8-4-19)$$

$$k_y'' = \pm \tau_x \quad (8-4-20)$$

So, if we can expressed the wavevector  $k$  at the point of stationary phase in terms of  $k_y'$  and  $k_y''$ , the radiation field can be found as a function of  $(x,t)$ . In general, it is difficult to do this and numerical method may have to be used. For some simple cases where the DWS of the medium is of the form (8-4-4), this can be done by a method similar to that in § 3.3. Definiate (8-4-4) :

$$2k_y k_y' = 2Ak_z + Ck_x + D \quad (8-4-21)$$

$$2k_y k_y'' = 2Bk_x + Ck_z + E \quad (8-4-22)$$

By these two equations, a linear relation of  $k_x$  and  $k_z$  can be

found :

$$k_z = \xi k_x + \eta \quad (8-4-23)$$

where

$$\xi = (2B\dot{k}_y' - C\dot{k}_y)/(2A\dot{k}_y' - C\dot{k}_y) \quad (8-4-24)$$

$$\eta = (E\dot{k}_y' - D\dot{k}_y)/(2A\dot{k}_y' - C\dot{k}_y) \quad (8-4-25)$$

Substituting (8-4-23) and (8-4-4) into (8-4-21), we get a second order equation of  $k_x$  :

$$k_x^2 [4CA\xi^2 + B + C\xi] - (2A\xi + C)^2 + k_x [4(2A\xi\eta + C\eta + D\xi + E) - 2(2A\xi + C)(2A\eta + D)] + 4(A\eta^2 + D\eta + F) - (2A\eta + D)^2 = 0$$

So,  $k_x$ , and thus  $k_z$  and  $k_y$ , can be expressed in terms of  $k_y'$  and  $\dot{k}_y$  easily.

Note that the above discussion can be extended to the case of a general quadratic DWS :

$$k_y^2 + k_y(Gk_z + Hk_x + I) = Ak_z^2 + Bk_x^2 + Ck_z k_x + Dk_z + Ek_x + F \quad (8-4-26)$$

since the left hand side can be written as :

$$\tilde{k}_y^2 - (Gk_z + Hk_x + I)^2/4$$

where

$$\tilde{k}_y \equiv k_y + (Gk_z + Hk_x + I)/2 \quad (8-4-27)$$

So, (8-2-26) can be rewritten as:

$$\tilde{k}_y^2 = \tilde{A}k_z^2 + \tilde{B}k_x^2 + \tilde{C}k_z k_x + \tilde{D}k_z + \tilde{E}k_x + \tilde{F} \quad (8-4-28)$$

where  $\tilde{A}, \dots, \tilde{F}$  are new constants which can be found in terms of  $A, \dots, I$ . Using

$$\tilde{k}_y' = k_y' + H/2 \quad (8-4-29)$$

$$\tilde{k}_y = \dot{k}_y + G/2 \quad (8-4-30)$$



Thus, by similar consideration as above,  $k_x$ ,  $k_y$  and  $k_z$  can be written as explicit functions of  $(x, t)$ . Then the radiation field can then be found as a function of space and time, where  $\bar{\Gamma}$  and  $(\frac{\partial \mathcal{D}}{\partial k} + \frac{\partial \mathcal{D}}{\partial \omega} V)$  can still be found by (8-1-22), (8-1-15), and (8-2-16). Since the explicit function form is quite lengthy, it is not given here. However, it can be written out without much difficulty.

## CHAPTER IX

### CERENKOV RADIATION IN AN UNIAXIAL COLD PLASMA

#### 9.1 Dispersion relation of an uniaxial cold plasma

An uniaxial cold plasma is an cold magneto plasma with very large external magnetic field  $B = B \hat{z}$ . In this case, the quantities defined in § 3.2.5 become :

$$D = 0 \quad (9-1-1)$$

$$S = R = L = 1 \quad (9-1-2)$$

We see that the dielectric tensor is simplified to

$$\bar{\epsilon} = \bar{I} - (\omega_p/\omega)^2 \hat{z} \hat{z} \quad (9-1-3)$$

giving,

$$\bar{A} = (\omega^2 - c^2 k^2) \bar{I} - \omega_p^2 \hat{z} \hat{z} + c^2 k k \quad (9-1-4)$$

$$D = (\omega^2 - c^2 k^2)(\omega^2 - \omega_p^2)[\omega^2 - c^2 k_z^2 - c^2 k_\rho^2 / (1 - \frac{\omega_p^2}{\omega^2})] \quad (9-1-5)$$

$$\bar{\Gamma} = (\omega^2 - c^2 k^2)[(\omega^2 - \omega_p^2) \bar{I} + \omega_p^2 \hat{z} \hat{z} - c^2 k k] - \omega_p^2 c^2 k_\rho^2 \hat{\phi} \hat{\phi} \quad (9-1-6)$$

Equation (9-1-5) shows that there are three modes :

(p) :  $\omega^2 = \omega_p^2$ , a non-propagating mode

(i) :  $\omega^2 = c^2 k^2$ , the same dispersion relation as vacuum

$$(a) : \frac{c^2 k_\rho^2}{\omega^2 - \omega_p^2} + \frac{c^2 k_z^2}{\omega^2} = 1, \text{ an anisotropic mode} \quad (9-1-7)$$

For (i) mode

$$\frac{\partial D}{\partial k} = 2c^4 k_\rho^2 \frac{\omega_p^2}{\omega^2} k \quad (9-1-8)$$

$$\frac{\partial D}{\partial \omega} = -2c^2 k_\rho^2 \frac{\omega_p^2}{\omega^3} \omega \quad (9-1-9)$$

and, for the (a) mode ,

$$\frac{\partial D}{\partial k} = 2c^2 k_\rho^2 (\omega^2 - c^2 k^2) [\omega_p^2 k_z \hat{z} - \omega^2 k] \quad (9-1-10)$$

$$\frac{\partial D}{\partial \omega} = 2\omega(\omega^2 - c^2 k^2) [2\omega^2 - \omega_p^2 - c^2 k^2] \quad (9-1-11)$$

$\bar{\Gamma}$  can be rewritten as :



$$(i) : \bar{\Gamma} = -c^2 k_\rho^2 \omega_p^2 \hat{\phi} \hat{\phi} \quad (9-1-12)$$

$$(a) : \bar{\Gamma} = (\omega^2 - c^2 k^2) [(\omega^2 - \omega_p^2) \hat{\rho} \hat{\rho} + \omega^2 \hat{z} \hat{z} - c^2 k k] \quad (9-1-13)$$

So, these modes have similar properties as those in the non-dispersive uniaxial medium discussed in chapter VIII.

For a radiating source moving at constant  $V \hat{z}$ , the DWS of the (i) mode is the same as that of vacuum, i.e. (3-2-2).

For the (a) mode, the DWS is :

$$\begin{aligned} c^2 k_\rho^2 &= (\omega^2 - \omega_p^2)(\omega^2 - c^2 k_z^2) / \omega^2 \\ &= [(\frac{\omega}{\gamma} + k_z V)^2 - \omega_p^2] [(\frac{\omega}{\gamma} + k_z V)^2 - c^2 k_z^2] / (\frac{\omega}{\gamma} + k_z V)^2 \end{aligned} \quad (9-1-14)$$

Since it is not of the form (3-4-1), the stationary phase point value of  $k, \omega$  is hard to expressed in  $\tau$  explicitly, and thus the E field can not be found as function of space-time. Although we can still use (3-3-8) - (3-3-10) to find  $K$  and find  $k, \omega$  for a given  $(x, t)$  numerically, it will not be done here and will be left to the general discussion of cold magneto cold plasma in the next two chapters.

## 9.2 Cerenkov radiation for $V$ along $B$

One special case that the DWS reduces to the simple form of (3-4-1) is the Cerenkov radiation with current density :

$$J(k, \omega) = 2\pi q V \hat{z} \delta(\omega - k_z V - \omega) \quad (9-2-1)$$

Cerenkov radiation does not exist in the (i) mode since there is no intersection between  $\omega^2 = c^2 k^2$  and  $\omega = k_z V$  as that is vacuum. By (9-1-14), DWS of the (a) mode for Cerenkov radiation is :

$$\gamma^2 \beta^2 \frac{c^2 k_\rho^2}{\omega_p^2} + \beta^2 \frac{c^2 k_z^2}{\omega_p^2} = 1 \quad (9-2-2)$$

which is clearly an ellipsoid centered at the origin and with  $\hat{z}$  as its long axis , see Fig. 9-1 .

The group velocity can be calculated by (9-1-10) and (9-1-11) :

$$\mathbf{V}_E = c\beta k \frac{\gamma^2 k_\rho \hat{z} - k_z \hat{\rho}}{\gamma^2 k_\rho^2 + \frac{1}{\gamma^2} k_z^2} \quad (9-2-3)$$

where (9-2-2) has been used. Note that  $\hat{N} \parallel \mathbf{V}_E - \mathbf{V} \parallel \mathbf{x} - \mathbf{V}t$ ,

$$\mathbf{V}_E - \mathbf{V} = c\beta \frac{-k_z k_\rho \hat{\rho} - (k_z/\gamma)^2 \hat{z}}{\gamma^2 k_\rho^2 + \frac{1}{\gamma^2} k_z^2} \quad (9-2-4)$$

these vectors are drawn at Fig. 9-1 . So, we see that there are two stationary phase points for every observation point :

$$\mathbf{k} = -\alpha k_\rho \hat{\rho} + k_z \hat{z} \quad (9-2-5)$$

where  $\alpha = 1$  corresponds to the upper branch  $k_+$ ,  $\alpha = -1$  corresponds to the  $k_-$  and  $\hat{\rho}$  is now defined by the observation direction  $\mathbf{x} - \mathbf{V}t = \rho \hat{\rho} + (Vt-z)\hat{z}$ . Also, by (3-4-3) - (3-4-12) :

$$k_z = \sigma \gamma \omega_p \tau / \beta c (1 + \gamma^2 \tau^2)^{1/2} \quad (9-2-6)$$

$$k_\rho = \omega_p / \gamma \beta c (1 + \gamma^2 \tau^2)^{1/2} \quad (9-2-7)$$

$$\omega = \sigma \gamma \omega_p \tau / (1 + \gamma^2 \tau^2)^{1/2} \quad (9-2-8)$$

$$k = \frac{\omega_p}{\gamma \beta c} \left[ \frac{1 + \gamma^2 \tau^2}{1 + \tau^2} \right]^{1/2} \quad (9-2-9)$$

$$\lambda_{kz} = \sigma \frac{\beta c}{\gamma \omega_p} \left[ \frac{1 + \gamma^2 \tau^2}{1 + \tau^2} \right]^{3/2} \quad (9-2-10)$$

$$\lambda_\phi = \sigma \frac{\beta c \gamma}{\omega_p} \left[ \frac{1 + \gamma^2 \tau^2}{1 + \tau^2} \right]^{1/2} \quad (9-2-11)$$

$$K = \left[ \frac{\beta c}{\omega_p} \frac{1 + \gamma^2 \tau^2}{1 + \tau^2} \right]^2 \quad (9-2-12)$$

Note that  $\tau$  must be larger than zero as can be easily seen from Fig. 9-1 . Substitute these into (9-2-3) and (9-2-4) :

$$\mathbf{V}_E = \frac{c\beta (\hat{z} - \sigma \tau \hat{\rho})}{1 + \tau^2} \quad (9-2-13)$$



$$V_E = c\beta / (1 + \tau^2)^{1/2} \quad (9-2-14)$$

$$V_E - V = -c\beta\tau(\sigma \hat{\rho} + \tau \hat{z}) / (1 + \tau^2) \quad (9-2-15)$$

$$|V_E - V| = c\beta\tau / (1 + \tau^2)^{1/2} \quad (9-2-16)$$

By (9-2-14), we know that the group velocity of the radiation is always smaller than the velocity of the charge, although the phase velocity may be larger than  $V$  :

$$\frac{\omega}{k} = c\gamma^2\beta / (1 + \gamma^4\tau^2)^{1/2} \quad (9-2-17)$$

Using (9-1-11) and (9-2-16) :

$$\left( \frac{\partial \mathcal{D}}{\partial \mathbf{k}} + \frac{\partial \mathcal{D}}{\partial \omega} \mathbf{V} \right)_N = 2\omega(\omega^2 - c^2 k^2) \omega_p^2 c \tau (1 + \tau^2)^{1/2} / \beta (1 + \gamma^2 \tau^2) \quad (9-2-18)$$

Substitute into (2-20) and note that

$$A^\sigma = i\sigma \quad (9-2-19)$$

we find,

$$E(\mathbf{x}, t) = \sum_{\sigma} \frac{i\sigma q \omega_p (\mathbf{x} - \mathbf{V}t)}{c\beta \rho^2 (1 + \gamma^2 \tau^2)} e^{i(\mathbf{k}_\sigma \cdot \mathbf{x} - \omega_\sigma t)} \quad (9-2-20)$$

since  $\mathbf{k}_+ = -\mathbf{k}_-$  and  $\omega_+ = \omega_-$ ,

$$E(\mathbf{x}, t) = -2 \frac{q \omega_p (\mathbf{x} - \mathbf{V}t)}{c\beta \rho^2 (1 + \gamma^2 \tau^2)} \sin(\mathbf{k} \cdot \mathbf{x} - \omega t) \quad (9-2-20)$$

$$\begin{aligned} B(\mathbf{x}, t) &= c \mathbf{k} \times E(\mathbf{x}, t) / \omega \\ &= -2 \frac{q \omega_p \hat{\phi}}{c\rho (1 + \gamma^2 \tau^2)} \sin(\mathbf{k} \cdot \mathbf{x} - \omega t) \end{aligned} \quad (9-2-21)$$

where  $\mathbf{k}$ ,  $\omega$  are now  $\mathbf{k}_+$  and  $\omega_+$ . The Poynting vector is :

$$\begin{aligned} \langle S(\mathbf{x}, t) \rangle &= \frac{c}{4\pi} \langle \mathbf{E} \times \mathbf{B} \rangle \\ &= \frac{c}{2\pi\beta} \left[ \frac{q \omega_p}{c\rho (1 + \gamma^2 \tau^2)} \right]^2 (\hat{z} + \tau \hat{\rho}) \end{aligned} \quad (9-2-22)$$

Note that the direction of  $S$  is the same as that of the group velocity, but is different from that of  $\mathbf{k}$ . The time-averaged Poynting vector can also be calculated by (2-26) and using

$$|\mathbf{x} - \mathbf{V}t'|^2 = \frac{V_E^2 |\mathbf{x} - \mathbf{V}t|^2}{|\mathbf{V}_E - \mathbf{V}|^2} = \frac{\rho^2(1 + \tau^2)}{\tau^2}, \quad (9-2-23)$$

then,

$$\langle S \rangle = \frac{q^2 \omega_p^2}{8\pi\beta c \rho^2 (1 + \gamma^2 \tau^2)^2} (\hat{\mathbf{z}} + \tau \hat{\rho}) \quad (9-2-24)$$

The factor of four difference between (9-2-22) and (9-2-24) is due to the fact that we have only calculated one branch (one  $\alpha$ ) in (9-2-24), so the actual value should be four time larger.

The power received at  $(\mathbf{x}, t)$  per unit solid angle subtended at the retarded position is :

$$\frac{dP}{d\Omega}(\mathbf{x}, t) = |\mathbf{x} - \mathbf{V}t'|^2 \langle S \rangle = \frac{q^2 \omega_p^2 (1 + \tau^2)^{3/2}}{2\pi\beta c \tau^2 (1 + \gamma^2 \tau^2)^2} \quad (9-2-25)$$

The power emitted per unit solid angle can be found by (2-30) and (2-31). First, we note that :

$$V_E = c\beta/(1 + \tau^2)^{1/2} = c\beta \cos\theta \quad (9-2-26)$$

where  $\theta$  is the angle between  $\mathbf{V}_E$  and  $\mathbf{V}$ , so

$$f = \tan^2\theta = \tau^2 \quad (9-2-27)$$

$$\frac{dP'}{d\Omega} = f \frac{dP}{d\Omega} = \frac{q^2 \omega_p^2 (1 + \tau^2)^{3/2}}{2\pi\beta c (1 + \gamma^2 \tau^2)^2} \quad (9-2-28)$$

The total radiated power can then be calculated by putting  $\tau = \tan\theta$  and perform the integration :

$$W = \frac{q^2 \omega_p^2}{2\pi\beta c} \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin\theta d\theta \frac{(1 + \tan^2\theta)^{3/2}}{(1 + \gamma^2 \tan^2\theta)^2} \quad (9-2-29)$$

where the limit of  $\theta$  is taken as 0 to  $\pi/2$  since there is no backward radiation in this case. The integration can be easily performed by changing variable  $y = \cos^2\theta$  :



$$W = q^2 \omega_p^2 / 2c\beta\gamma^2 \quad (9-2-30)$$

Based on the above results, some features can be discussed :

(1) The power received (9-2-25) tends to infinity when  $\tau = 0$  , i.e. on the  $z$ -plane at the time  $t = z/V$  where the charge is also on this plane. However,  $\langle S \rangle$  is finite. This "contradiction" can be resolved by noting that the energy recieved on the  $z = Vt$  plane is of flow direction parallel to  $\hat{z}$ . This means that  $dP/d\Omega$  for  $\Delta\Omega$  at  $\theta = 0$  is actually recieved by the whole plane  $z = Vt$ , so the divergence of  $dP/d\Omega$  in this case is not surprising. Also, the emitted power per unit solid angle  $dP'/d\Omega$  (9-2-28)) is finite for  $\tau = 0$  .

(2) All quantities  $E$ ,  $\langle S \rangle$ ,  $dP/d\Omega$ ,  $dP'/d\Omega$ ,  $W$  tend to infinity for  $\beta \rightarrow 0$  . This result is physically unacceptable. The reason is that in the cold plasma modal, we have assumed that

$$\omega/k \gg V_{th} \quad (9-2-31)$$

where  $\omega/k$  is the phase velocity of the wave and  $V_{th}$  is the typical thermal velocity of the particles in the plasma. However, From (9-2-17) we see that  $\omega/k \rightarrow 0$  as  $\beta \rightarrow 0$  . Or we know that cold plasma model breakdown near the resonant frequency , i.e.  $ck/\omega \rightarrow \infty$  ; and for  $\beta \rightarrow 0$  in the Cerenkov radiation, the DWS is just on the frequency near the resonance. So, the cold plasma modal in this case is not valid, so the result calculated in this case does not correspond to the actual physical situation. For large  $\beta$  such that (9-2-31) is satisfied, the above equations should be correct.

(3) For  $\beta \rightarrow 1$ ,  $E$ ,  $\langle S \rangle$ ,  $dP'/d\Omega$  tends to zero except for  $\tau$

$= 0$  , i.e. for the relativistic case, the Cerenkov radiation concentrated near the plane  $z = Vt$ . However, the total radiated power  $W$  (9-2-30) also tends to zero in this case. So, for highly relativistic charge, the energy loss due to Cerenkov radiation is less important. This can be seen also by the fact that the DWS reduces to a line when  $\beta \rightarrow 1$  .

(4) For  $\omega_p \rightarrow 0$ , all the radiation field quantities tend to zero. This result is expected since there is no Cerenkov radiation for vacuum.

### 9.3 Compare the radiation power found by J•E method

To give a check to (9-2-30), we now calculate the radiation power by the J•E method :

$$W = - \int dx J(x,t) \cdot E(x,t) \quad (9-3-1)$$

Applying Parseval's theorem :

$$W = - \frac{1}{(2\pi)^3} \int dk J^*(k,t) \cdot E(k,t) \quad (9-3-2)$$

where  $J$  can be calculated from (9-2-1), and  $E$  can be found by the general equation (2-10) :

$$J(k,t) = qV \exp -i(k \cdot V + i\nu)t \quad (9-3-3)$$

$$E(k,t) = -4\pi i \int_{-\infty}^{\infty} \frac{d\omega}{\mathcal{D}(k,\omega)} \frac{\omega \bar{\Gamma}(k,\omega) \cdot qV}{\delta(\omega - k_z V - i\nu)} e^{-i\omega t} \quad (9-3-4)$$

$$W = \frac{2iq^2}{(2\pi)^2} \int d\phi \int_0^\infty k_\rho dk_\rho \int_{-\infty}^{\infty} d\omega \frac{\omega V^2 \Gamma_{zz}}{\mathcal{D}} \delta(\omega - k_z V - i\nu) e^{-i(\omega - k_z V - i\nu)t} \quad (9-3-5)$$

where  $\Gamma_{zz}$  and  $\mathcal{D}$  can be found by (9-1-5) and (9-1-6) :

$$\Gamma_{zz} / \mathcal{D} = -(\omega^2 - c^2 k_z^2) / c^2 \omega^2 (k_\rho^2 - \xi) \quad (9-3-6)$$

where

$$\xi \equiv (\omega^2 - \omega_p^2)(\omega^2 - c^2 k_z^2) / \omega^2$$

The integration over  $\phi$  and  $k_z$  can be performed easily :



$$W = - \frac{iq^2}{2\pi c\gamma^2\beta} \int_0^\infty dk \left( \frac{k^2}{\rho} \right) \int_{-\infty}^\infty d\omega \frac{\omega}{k^2_{\rho} - \mathcal{R}e \xi - i\mathcal{I}m \xi} \quad (9-3-7)$$

where

$$\mathcal{R}e \xi = - (\omega^2 - \omega_p^2) / \gamma^2\beta^2 \quad (9-3-8)$$

$$\mathcal{I}m \xi = \nu(\omega^2 - \omega_p^2) / \omega\beta^2 \quad (9-3-9)$$

Since  $\mathcal{R}e \xi$  is even function of  $\omega$ , and  $\mathcal{I}m \xi$  is odd function of  $\omega$ , we can put  $\omega \rightarrow -\omega$  for  $\omega < 0$  :

$$W = - \frac{iq^2}{2\pi c\gamma^2\beta} \int_0^\infty \omega d\omega \left\{ \int_0^\infty \frac{dx}{x - \mathcal{R}e \xi - i\mathcal{I}m \xi} - \int_0^\infty \frac{dx}{x - \mathcal{R}e \xi + i\mathcal{I}m \xi} \right\} \quad (9-3-10)$$

where  $x \equiv k^2_{\rho}$ . So, if  $\mathcal{R}e \xi > 0$ , then there is a pole on the  $x$  axis and gives the dominant contribution. This require  $\omega < \omega_p$  and thus  $\mathcal{I}m \xi \rightarrow 0^-$  as  $\nu \rightarrow 0^+$ . So, the  $x$  - integration can be performed in the complex -  $x$  plane using the contour as Fig. 9-2. We then have,

$$W = \frac{(-2\pi i)iq^2}{2} \int_0^{\omega_p} \omega d\omega = q^2\omega_p^2 / 2c\beta\gamma^2 \quad (9-3-11)$$

which is just (9-2-30).

Note that the divergence of  $W$  as  $\beta \rightarrow 0$  still exist in the J•E method. However, in the above deviation,  $\beta \neq 0$  is assumes since  $\xi$  is infinite for  $\beta = 0$ .

## CHAPTER X

### RADIATION IN A COLD MAGNETOPLASMA

#### 10.1 Dispersion relation of a cold magnetoplasma

The dielectric tensor of a cold magneto plasma is :

$$\bar{\epsilon} = S \bar{I} + (P - S) \hat{z}\hat{z} + iD \hat{z} \times \bar{I} \quad (10-1-1)$$

where we have used the definition (3-2-16) - (3-2-21) and the external magnetic field is along  $\hat{z}$ . Note that

$$\hat{z} \times \bar{I} = \hat{\phi}\hat{\rho} - \hat{\rho}\hat{\phi} = \hat{y}\hat{x} - \hat{x}\hat{y} \quad (10-1-2)$$

$$\hat{\rho} = \cos\phi \hat{x} + \sin\phi \hat{y}, \quad \hat{\phi} = -\sin\phi \hat{x} + \cos\phi \hat{y} \quad (10-1-3)$$

$$\hat{x} = \cos\phi \hat{\rho} - \sin\phi \hat{\phi}, \quad \hat{y} = \sin\phi \hat{\rho} + \cos\phi \hat{\phi} \quad (10-1-4)$$

So,

$$\bar{\Delta} = (\omega^2 S - c^2 k^2) \bar{I} + \omega^2 (P - S) \hat{z}\hat{z} + c^2 k k + i\omega^2 D \hat{z} \times \bar{I} \quad (10-1-5)$$

$$\begin{aligned} \bar{\Gamma} &= (\omega^2 S - c^2 k^2) [\omega^2 P \bar{I} - \omega^2 (P - S) \hat{z}\hat{z} + c^2 k k] + \omega^2 (P - S) c^2 k_{\rho}^2 \hat{\phi}\hat{\phi} \\ &\quad - \omega^4 D^2 \hat{z}\hat{z} - i\omega^2 D [(c^2 P - c^2 k_{\rho}^2) \hat{z} + c^2 k_{\rho} k_z \hat{\rho}] \times \hat{z} \end{aligned} \quad (10-1-6)$$

Note that

$$\hat{\rho} \times \bar{I} = \hat{z}\hat{\phi} - \hat{\phi}\hat{z} = \sin\phi (\hat{x}\hat{y} - \hat{y}\hat{x}) + \cos\phi (\hat{z}\hat{y} - \hat{y}\hat{z}) \quad (10-1-7)$$

(10-1-5) and (10-1-6) may be more concrete if we write it in matrix form of  $\hat{\rho}, \hat{\phi}, \hat{z}$  :

$$\bar{\Delta}(k, \omega) = \begin{bmatrix} \omega^2 S - c^2 k_z^2 & -i\omega^2 D & c^2 k_{\rho} k_z \\ i\omega^2 D & \omega^2 S - c^2 k^2 & 0 \\ c^2 k_{\rho} k_z & 0 & \omega^2 P - c^2 k_{\rho}^2 \end{bmatrix} \quad (10-1-8)$$

$$\bar{\Gamma} = \begin{bmatrix} (\omega^2 S - c^2 k^2)(\omega^2 P - c^2 k_{\rho}^2) & i\omega^2 D(\omega^2 P - c^2 k_{\rho}^2) & -c^2 k_{\rho} k_z (\omega^2 S - c^2 k^2) \\ -i\omega^2 D(\omega^2 P - c^2 k_{\rho}^2) & \omega^4 S P - \omega^2 S c^2 k_{\rho}^2 - \omega^2 P c^2 k_z^2 & i\omega^2 D c^2 k_{\rho} k_z \\ -c^2 k_{\rho} k_z (\omega^2 S - c^2 k^2) & -i\omega^2 D c^2 k_{\rho} k_z & (\omega^2 S - c^2 k^2)(\omega^2 S - c^2 k_z^2) - \omega^4 D^2 \end{bmatrix} \quad (10-1-9)$$

they can also be rewritten in  $\hat{x}, \hat{y}, \hat{z}$  :



$$\bar{\Delta} = \begin{bmatrix} \omega^2 S - c^2 k_\rho^2 \sin^2 \phi - c^2 k_z^2 & -i\omega^2 D + c^2 k_\rho^2 \sin \phi \cos \phi & c^2 k_\rho k_z \cos \phi \\ i\omega^2 D + c^2 k_\rho^2 \sin \phi \cos \phi & \omega^2 S - c^2 k_\rho^2 \cos^2 \phi - c^2 k_z^2 & c^2 k_\rho k_z \sin \phi \\ c^2 k_\rho k_z \cos \phi & c^2 k_\rho k_z \sin \phi & \omega^2 P - c^2 k_\rho^2 \end{bmatrix} \quad (10-1-10)$$

$$\bar{\Gamma} = \begin{bmatrix} a_1 + a_2 \cos \phi & a_2 \sin \phi \cos \phi + ia_3 & a_5 \cos \phi - ia_4 \sin \phi \\ a_2 \sin \phi \cos^2 \phi - ia_3 & a_1 + a_2 \sin^2 \phi & a_5 \sin \phi + ia_4 \cos \phi \\ a_5 \cos \phi + ia_4 \sin \phi & a_5 \sin \phi - ia_4 \cos \phi & a_6 \end{bmatrix} \quad (10-1-11)$$

where

$$a_1 \equiv \Gamma_{\phi\phi} = \omega^4 SP - \omega^2 S c^2 k_\rho^2 - \omega^2 P c^2 k_z^2 \quad (10-1-12)$$

$$a_2 \equiv \Gamma_{\rho\rho} - \Gamma_{\phi\phi} = c^2 k_\rho^2 (c^2 k^2 - \omega^2 P) \quad (10-1-13)$$

$$a_3 \equiv |\Gamma_{\rho\phi}| = \omega^2 D (\omega^2 P - c^2 k_\rho^2) \quad (10-1-14)$$

$$a_4 \equiv |\Gamma_{\phi z}| = \omega^2 D c^2 k_\rho k_z \quad (10-1-15)$$

$$a_5 \equiv \Gamma_{\rho z} = c^2 k_\rho k_z (c^2 k^2 - \omega^2 S) \quad (10-1-16)$$

$$a_6 \equiv \Gamma_{zz} = (\omega^2 S - c^2 k_\rho^2)(\omega^2 S - c^2 k_z^2) - \omega^4 D^2 \quad (10-1-17)$$

The dispersion function  $\mathcal{D}$  is given by (3-2-22). Because of (3-2-25), we can rewrite  $\mathcal{D}$  as

$$\mathcal{D} = c^4 \omega^2 (k_\rho^2 - \xi_+) (k_\rho^2 - \xi_-) \quad (10-1-18)$$

where  $\xi_\pm$  are functions of  $\omega^2$  and  $k_z^2$ :

$$\xi_\pm = \frac{1}{2c^2} \left\{ \omega^2 \left( P + \frac{RL}{S} \right) - \left( 1 + \frac{P}{S} \right) c^2 k_z^2 \pm \left[ \left[ \omega^2 \left( P + \frac{RL}{S} \right) - \left( 1 + \frac{P}{S} \right) c^2 k_z^2 \right]^2 - \frac{4P}{S} \left[ \omega^4 RL - (2\omega^2 S - c^2 k_\rho^2) c^2 k_z^2 \right] \right]^{1/2} \right\} \quad (10-1-19)$$

Since  $\mathcal{D}$  is cylindrically symmetric about  $\hat{z}$ , so the dispersion relation  $\mathcal{D} = 0$  can be represented by a surface in three dimensional space  $(\omega, k_\rho, k_z)$  with its shape depends on parameters  $\omega_{pe}, \omega_{pi}, \Omega_e, \Omega_i$  which is given by (3-2-20) and (3-2-21). For simplicity, we consider the case of a hydrogen

plasma, i.e. putting

$$Z = 1, n_e = n_i \text{ and } m_i = 1836 m_e \equiv M m_e \quad (10-1-20)$$

define :

$$X \equiv \Omega_e / \omega, \quad Y \equiv (\omega_{pe} / \omega)^2 \quad (10-1-21)$$

then (3-2-16) - (3-2-19) can be rewritten as :

$$S = 1 - \left[ \frac{1}{1 - X^2} + \frac{M}{M^2 - X^2} \right] Y \quad (10-1-22)$$

$$P = 1 - \left( 1 + \frac{1}{M} \right) Y \quad (10-1-23)$$

$$D = \left[ \frac{1}{M^2 - X^2} - \frac{1}{1 - X^2} \right] X Y \quad (10-1-24)$$

$$R = 1 - \left[ \frac{1}{1 - X} + \frac{1}{M + X} \right] Y \quad (10-1-25)$$

$$L = 1 - \left[ \frac{1}{1 + X} + \frac{1}{M - X} \right] Y \quad (10-1-26)$$

Since  $M$  is a large parameter, so it can be seen that the contribution from ion is important only for low frequency, especially for  $X \sim M$ , i.e.  $\omega \sim \Omega_i$ .

We can study the dispersion surface by looking at its cross-sectional shape at a fixed  $\theta$ . The surface is then reduces to a curve  $\omega = \omega(k)$  which can be plotted by using (3-2-23). In a hydrogen cold magnetoplasma, we see that the shape of the dispersion surface is just determined by the ratio of  $\omega_{pe}$  and  $\Omega_e$ . We define :

$$a \equiv (\omega_{pe} / \Omega_e)^2 \quad (10-1-27)$$

$$Y = a X^2 \quad (10-1-28)$$

The curves  $\omega = \omega(k, \theta)$  for the two cases :  $a < 1$  and  $a > 1$  are plotted respectively in Fig 10-1's and Fig 10-2's for several values of  $\theta$ . Some points are noted :

(1) There is actually another branch coming out from the origin ( $\omega = k = 0$ ) and tends to infinity ( $ck/\omega \rightarrow \infty$ ) near  $\Omega_i$  as drawn in the schematic plot of the dispersion surface



(Fig 10-3). However, since  $\Omega_i \ll \Omega_e$  and the scale of Fig 10-1 and 10-2 is linear, this branch can not be resolved out from the axis  $\omega = 0$ .

(2) For general  $\theta$ , there should be three cut-off ( $ck/\omega = 0$ ) and three resonances ( $ck/\omega \rightarrow \infty$ ). The cut-off frequencies are determined by  $R = 0$ ,  $L = 0$  and  $P = 0$  ( $\omega = \omega_p$ ) and the resonance frequencies are determined by the following relation as can be seen from (3-2-23) :

$$P / S = -\tan^2 \theta \quad (10-1-29)$$

However, for the special case  $\theta = 0$ , the number of both kinds of frequencies reduce to two because now  $\omega = \omega_p$  corresponds to both cut-off and resonance frequency (Note that it is a non-propagating mode). For  $\theta = \pi/2$ , the resonance frequencies are given by  $S = 0$  and there are two  $\omega$  values in this case. Note also that From (10-1-29), resonance only occurs in regions that  $S$  and  $P$  having opposite signs.

(3) For high frequency  $\omega \gg \Omega_e$ , the dispersion relation tends to that of vacuum, i.e.  $\omega = ck$ .

(4) The dispersion surface depends on  $\omega^2$ ,  $k_\rho^2$  and  $k_z^2$ , so the graphs for  $\omega < 0$  and  $\theta > \pi/2$  can be easily produced from Fig 10-1's and 10-2's by a symmetric consideration.

We can study the dispersion surface also by looking at its cross-sectional view of fixed  $\omega$ , i.e. the wave-vector surface (i.e. DWS with  $\beta = 0$ ). From Fig's 10-1 to 10-3, we see that the topology of the wave-vector surface changes whenever  $\omega$  gets across regions separated by zeros and poles of  $S$ ,  $P$ ,  $D$ ,  $R$ ,  $L$ . Since these parameters depend on  $X$ ,  $Y$ , we can draw the curves of these zeros and poles on the  $X$ ,  $Y$  plane. These curves then separate the  $X$ ,  $Y$  plane into several regions so that the topology of the wave-vector surface is



different from region to region. This graph is known as Clemmow - Mullaly - Allis (CMA) diagram (e.g. Stix, 1962). By (10-1-22) - (10-1-26), we know that :

$$P = 0 : Y = 1/(1 + 1/M) \cong 1 \quad (10-1-30)$$

$$S = 0 : Y = \frac{(1 - X^2)[1 - (X/M)^2]}{(1 + 1/M)(1 - X^2/M)} \quad (10-1-31)$$

$$R = 0 : Y = \frac{(1 - X)(1 + X/M)}{1 + 1/M} \quad (10-1-32)$$

$$L = 0 : Y = \frac{(1 + X)(1 - X/M)}{1 + 1/M} \quad (10-1-33)$$

$$S, D, R = \infty : X = 1 \quad (10-1-34)$$

$$S, D, L = \infty : X = M \quad (10-1-35)$$

These equations are plotted in the X-Y plane in a log-log scale as shown in Fig 10-4. The dot • on the graph makes it easy to identify what region correspond to a given (X,Y). We have also plotted the line  $Y = aX^2$  on the graph for  $a = 0.3$  and 1.2 i.e. the values used in Figs.10-1 and 10-2. Two curves are also plotted :

$$R = -P : Y = \frac{2(1 - X)(M + X)}{(M + 1)[1 + (1 - X)(1 + X/M)]} \quad (10-1-36)$$

$$L = -P : Y = \frac{2(1 + X)(M - X)}{(M + 1)[1 + (1 + X)(1 - X/M)]} \quad (10-1-37)$$

The reason for drawing these curves will be explained later in chapter XI.

Wave-vector surfaces for different X,Y in the CMA diagram are plotted in Fig. 10-5 by using (3-2-23) - (3-2-25). Note that the unit of k is  $\omega/c$ , i.e. we are plotting the refractive index  $ck/\omega$ . Some points can be discussed from these graphs :

(1) In Lai and Chan's method for calculating the radiation field , i.e. (2-20), the Gaussian curvature K of the wave-vector surface surface is assumed to be non-zero.



However, there are several cases in the cold magnetoplasma that  $K = 0$ . One case is that the curve  $k_z$  vs  $k_\rho$  may have a maximum or a minimum  $k_z$ , i.e.  $dk_z/dk_\rho = 0$  or  $\tau = \infty$ , at  $k_\rho \neq 0$ . From (3-3-9), we see that  $\lambda_\phi = 0$  in this case, so  $K = 0$ . Another case is that there may be turning points on the  $k_z$ - $k_\rho$  curve, i.e.  $d^2k_\rho/dk_z^2 = 0$ . From (3-3-8),  $\lambda_{kz} = 0$  in this case, so  $K = 0$  again. Some typical wave-vector surfaces that belong to these two cases are drawn in Fig 10-6. When  $K = 0$ , (2-20) can not be used, so the radiation field should be calculated separately for these cases. For the case of  $\lambda_\phi = 0$ , since the surface is cylindrically symmetric, there exists a whole circular path on the wave-vector <sup>surface</sup> where  $K = 0$  and the group velocity is along the  $z$ -direction; the radiation along this direction is therefore enhanced (Lighthill, 1960; Giles, 1978). This effect will be discussed in chapter XI. For the case where  $\lambda_{kz} = 0$ , since only one point of surface is having  $K = 0$ , the enhancement of radiation field is not so great as that of the  $\lambda_\phi = 0$  case and so it will not be discussed in this thesis.

(2) For a cold magnetoplasma, there are at most two sheets of wave-vector surface. Also, for each  $k_z$ , there are at most two values of  $k_\rho$ ; and for each  $k_\rho$ , there are at most two values of  $k_z$ . These points can be seen from (3-2-23)-(3-2-25).

(3) At  $\theta = 0$ , the  $k$  value(s) is given by

$$n^2 = (ck/\omega)^2 = R \text{ or } L \quad (10-1-38)$$

(4) Whenever  $S$  and  $P$  having opposite signs, the wave-vector surface will have a resonance angle  $\theta_r$ , given by (10-1-29) so that  $k \rightarrow \infty$  at this angle and the surface exist only in  $\theta > \theta_r$  or  $\theta < \theta_r$ .



(5) For  $B \rightarrow 0$ ,  $R \cong L \cong P$ , so the wave-vector reduces to that of isotropic cold plasma as expected.

## 10.2 DWS for V along B

From the discussion in § 3.2.4, it has been shown that a DWS in case of a radiating source moving along the direction of the external magnetic field in a cold magnetoplasma can be expressed in the form of  $k_\rho$  as an explicit function of  $k_z$ . Since this functional form is quite complicated, it is hard to study it analytically. Instead, we shall study it by plotting the graph of  $k_\rho$  vs  $k_z$  using numerical computation. Some examples of these plots are given in Figs. 10-8 (a)-(f), and 10-9 (a)-(b), where we used  $\omega_c/c$  as the unit of  $k$ , i.e. we are plotting the graphs  $n_z$  vs  $n_\rho$  with

$$n \equiv ck/\omega_0 \quad (10-2-1)$$

and by (3-2-25) :

$$n_\rho^2 = \frac{(\frac{1}{\gamma} + n_z \beta)^2}{2} \left\{ \left( \frac{SP+RL}{S} \right) - \left( \frac{S+P}{S} \right) \frac{n_z^2}{(\frac{1}{\gamma} + n_z \beta)^2} \pm \left[ \left( \left( \frac{SP+RL}{S} \right) - \left( \frac{S+P}{S} \right) \frac{n_z^2}{(\frac{1}{\gamma} + n_z \beta)^2} \right)^2 - \frac{4P}{S} \left[ RL - \left[ 2S - \frac{n_z^2}{(\frac{1}{\gamma} + n_z \beta)^2} \right] \frac{n_z^2}{(\frac{1}{\gamma} + n_z \beta)^2} \right]^{1/2} \right\} \quad (10-2-2)$$

where  $S, P, D, R, L$  are still functions given by (10-1-29) - (10-1-33) but with  $X, Y$  now changed as :

$$X \rightarrow X / (\frac{1}{\gamma} + n_z \beta) = (\frac{\Omega_e}{\omega_0}) / (\frac{1}{\gamma} + n_z \beta),$$

$$Y \rightarrow Y / (\frac{1}{\gamma} + n_z \beta)^2 = (\frac{\omega_{pe}}{\omega_0})^2 / (\frac{1}{\gamma} + n_z \beta)^2, \quad (10-2-3)$$

or, we can say that the  $X, Y$  parameters change along the curve  $Y = a X^2$  as  $\beta$  changes. We have already seen that this curve is just a straight line in the CMA diagram, Fig. 10-4, with slope =  $\ln 2$ .



Note that we have assumed that  $\omega_0 \neq 0$ , so the above equations (10-2-1) cannot be applied to the case of Cerenkov radiation where  $\omega_0 = 0$ . Instead, we should use the general equation (3-2-19) in this case.

With the help of Figs. 10-8 and 10-9, some features of the DWS can be discussed:

(1) For small  $\beta$ , the DWS is just slightly different from the wave-vector surface for  $\beta = 0$ . In fact, it is just shifted a little upward. However, since the dispersion surfaces extend to infinity at three  $\omega$ -values (see Fig 10-1 to 10-3) so the Doppler plane must intersect these branches eventually for  $k_z \rightarrow \infty$  (note the Doppler plane extends to  $\omega < 0$  for  $k_z < 0$  and will intersect the dispersion surface there.). So, in principle, the DWS should consist of several branches, no matter how small  $\beta$  is. This effect can be seen clearer for the case of  $\beta \rightarrow 1$ , e.g. Fig 10-8e,f show the DWS for  $\beta = 0.9$ . It can be seen that there are three branches for  $k_z < 0$ . This effect makes the DWS much more complicated, especially for large  $\beta$ . For  $\beta \cong 0$ , although there are still many additional branches for large  $k_z$ , these branches correspond to the case of resonance where the cold plasma model is no longer valid. One example for this "resonance wave-vector surface" is the wave-vector surface of the uniaxial cold magnetoplasma ( $B \rightarrow \infty$ ) for the Cerenkov radiation which has been considered in chapter IX. From Fig. 9-1, we see that this surface becomes infinitely large for  $\beta \rightarrow 0$ . This effect makes the radiation power diverge for  $\beta \rightarrow 0$  as has been discussed in § 9-2.

(2) In the case of  $\beta = 0$ , the  $\theta = 0$  values of  $k$  are just those given by  $n^2 = R$  or  $n^2 = L$ ; but for DWS, there are



additional  $k$ -values given by  $P(k_z) = 0$  (Note that since  $\omega = \frac{\omega_0}{\gamma} + k_z V$ , so  $S, P, D, R, L$  are now functions of  $k_z$ ). The existence of these additional values may also be seen in Fig's 10-1 and 10-2, if we note that the Doppler plane can intersect the dispersion surface very near the plane  $P = 0$  for  $\theta \rightarrow 0$ . For an ordinary wave-vector surface, i.e. when the Doppler plane reduces to a constant  $\omega$  plane  $\beta = 0$ , this can not happen except for  $\omega = \omega_p$ . Some cases when this additional values appear have been pointed out in Fig. 10-8.

(3) There exists some cases, when the parameters  $X, Y$  are inside the region 5 of the CMA diagram, that there is no wave-vector surface when  $\beta = 0$ , but having DWS for finite  $\beta$ . This is because for  $\beta \neq 0$ ,  $\omega$  increases with  $k_z V$  such that the parameters  $X, Y$  move along the  $Y = aX^2$  line in the CMA diagram (as shown in Fig. 10-4). The  $X, Y$  values, initially inside region 5, will come out along the  $Y = aX^2$  line.

(4) The existence of  $\lambda_\phi = 0$  case, i.e. maximum or minimum of  $k_\rho(k_z)$  at  $k_\rho \neq 0$ , is of interest as has been discussed in § 10-1. These cases also exist in DWS for general  $\beta$  and actually more complicated. There are situations where we have  $\lambda_\phi \neq 0$  everywhere for  $\beta = 0$ , but  $\lambda_\phi$  could vanish, at some points on the DWS (Fig. 10-8c,d) for  $\beta \neq 0$ . On the contrary, there are cases <sup>that</sup>  $\lambda_\phi = 0$  exist for  $\beta = 0$  but becomes non-zero everywhere for a certain  $\beta$  (Fig. 10-9c,d). Fig. 10-9 show how the first case happens: When  $k_z$  increase or decrease,  $\omega$  increase or decrease according to the Doppler equation. Originally,  $X, Y$  are in the region where the  $\lambda_\phi = 0$  case does not exist. However, as  $\omega$  changes, the  $X, Y$  values enter to some region that  $\lambda_\phi = 0$  case can exist. The points on the DWS actually is the intersection of the Doppler plane



(represented by  $n_z = \text{constant}$  line in Fig. 10-9) and the ordinary wave-vector surface at the same  $\omega$  (which is also drawn in Fig. 10-9). So the crossing of regions as  $k_z$  increase or decrease brings in the case of  $\lambda_\phi = 0$ . Note also that there could be two sheets of DWS for a fixed  $\beta$ , each having a point where  $\lambda_\phi = 0$  (see the two sheets for  $\beta = 0.05$  in Fig. 10-8c).

### 10.3 Finding the points of stationary phase

In order to find the points of stationary phase of the DWS, the normal of the surface  $\hat{N} \parallel \mathbf{V}_E - \mathbf{V}$  should be first found Using (10-1-18) and rewrite (10-1-19) as :

$$\xi_{\pm} = \Lambda \pm \sqrt{\Theta} \quad (10-3-1)$$

where

$$\Theta \equiv \Lambda^2 - \Delta \quad (10-3-2)$$

$$\Lambda \equiv \frac{1}{2} \left[ \left( \frac{\omega}{c} \right)^2 \lambda_1 - (\lambda_2 + 1) k_z^2 \right] \quad (10-3-3)$$

$$\Delta \equiv \lambda_2 \left( \frac{\omega}{c} \right)^4 \lambda_3 - [2 \left( \frac{\omega}{c} \right)^2 S - k_z^2] k_z^2 \quad (10-3-4)$$

$$\lambda_1 \equiv P + (\lambda_3 / S) \quad (10-3-5)$$

$$\lambda_2 \equiv P / S \quad (10-3-6)$$

$$\lambda_3 \equiv RL \quad (10-3-7)$$

By direct differentiation :

$$\frac{\partial \mathcal{D}}{\partial \mathbf{k}} = \frac{\partial \mathcal{D}}{\partial k_\rho} \hat{\rho} + \frac{\partial \mathcal{D}}{\partial k_z} \hat{z} \quad (10-3-8)$$

$$\begin{aligned} \frac{\partial \mathcal{D}}{\partial k_\rho} &= 2c^4 \omega^2 k_\rho [(k_\rho^2 - \xi_+) + (k_\rho^2 - \xi_-)] \\ &= 4c^4 \omega^2 k_\rho (k_\rho^2 - \Lambda) \end{aligned} \quad (10-3-9)$$

$$\begin{aligned} \frac{\partial \mathcal{D}}{\partial k_z} &= -c^4 \omega^2 [(k_\rho^2 - \xi_+) \xi'_- + (k_\rho^2 - \xi_-) \xi'_+] \\ &= -c^4 \omega^2 (2k_\rho^2 \Lambda' - \Delta') \end{aligned} \quad (10-3-10)$$

where ' denotes  $\partial/\partial k_z$  and thus

$$\Lambda' = -(\lambda_2 + 1) k_z \quad (10-3-11)$$

$$\Delta' = 4\lambda_z k_z [k_z^2 - (\frac{\omega}{c})^2 S] \quad (10-3-12)$$

Finally,

$$\frac{\partial \mathcal{D}}{\partial k} = 2c^4 \omega^2 [2k_\rho (k_\rho^2 - \Lambda) \hat{\rho} + ((\lambda_z + 1)k_\rho^2 + 2\lambda_z [k_z^2 - (\frac{\omega}{c})^2 S])k_z \hat{z}] \quad (10-3-13)$$

Similarly ,

$$\frac{\partial \mathcal{D}}{\partial \omega} = -c^4 \omega^2 (2k_\rho \dot{\Lambda} - \dot{\Delta}) \quad (10-3-14)$$

where  $\dot{\phantom{x}}$  denotes  $\partial/\partial\omega$  and  $\mathcal{D} = 0$  has been used.

$$\dot{\Lambda} = \frac{\omega}{c^2} \lambda_1 + \frac{1}{2} (\frac{\omega}{c})^2 \dot{\lambda}_1 - \frac{1}{2} \dot{\lambda}_2 k_z^2 \quad (10-3-15)$$

$$\dot{\Delta} = \Delta(\dot{\lambda}_2/\lambda_2) + \lambda_2 [\frac{4\omega^3}{c^4} \dot{\lambda}_3 + (\frac{\omega}{c})^4 \dot{\lambda}_3 - 2(\frac{2\omega}{c^2} S + \frac{\omega^2}{c^2} \dot{S})k_z^2] \quad (10-3-16)$$

$$\dot{\lambda}_1 = \dot{P} + \dot{\lambda}_3/S - \lambda_3 \dot{S}/S^2 \quad (10-3-17)$$

$$\dot{\lambda}_2 = \dot{P}/S - P\dot{S}/S^2 \quad (10-3-18)$$

$$\dot{\lambda}_3 = \dot{R}L + R\dot{L} \quad (10-3-19)$$

$$\dot{S} = (S-1)\dot{Y}/Y - 2[\frac{1}{(1-X^2)^2} + \frac{M}{(M^2-X^2)^2}]X\dot{X}Y \quad (10-3-20)$$

$$\dot{P} = -(1 + \frac{1}{M}) \dot{Y} \quad (10-3-21)$$

$$\dot{R} = (R-1)\dot{Y}/Y - [\frac{1}{(1-X)^2} - \frac{1}{(M+X)^2}]X\dot{X}Y \quad (10-3-22)$$

$$\dot{L} = (L-1)\dot{Y}/Y + [\frac{1}{(1+X)^2} - \frac{1}{(M-X)^2}]X\dot{X}Y \quad (10-3-23)$$

$$\dot{X} = -\Omega_e/\omega^2 = -X/\omega \quad (10-3-24)$$

$$\dot{Y} = -2\omega_{pe}^2/\omega^3 = -2Y/\omega \quad (10-3-25)$$

So,  $\partial\mathcal{D}/\partial\omega$  can be found without much difficulty by these equations. Then the group velocity can be found by

$$\mathbf{V}_E = -\frac{\partial \mathcal{D}}{\partial \mathbf{k}} / \frac{\partial \mathcal{D}}{\partial \omega}$$

and the direction of  $\mathbf{V}_E - \mathbf{V}$  gives the normal direction of the DWS, which must be parallel to  $\mathbf{x} - \mathbf{V}t$  if it is the point of stationary phase.

In order to find the point of stationary phase, we must



calculated  $\partial k_\rho / \partial k_z$  with the DWS is written as

$$k_\rho^2 = \xi_\pm \quad (10-3-6)$$

Note that S, P, D, R, L are now functions of  $k_z$  through the  $\omega$  dependence  $\omega = (\omega_0/\gamma) + ck_z \beta$ . So

$$\tau = \frac{dk_\rho}{dk_z} = \frac{1}{2k_\rho} [\Lambda' + \dot{\Lambda}c\beta \pm \frac{\Theta' + \dot{\Theta}c\beta}{2\sqrt{\Theta}}] \quad (10-3-27)$$

where now the  $'$  denotes  $(\partial/\partial k_z)_{\omega=\text{constant}}$ , i.e. the quantities are given by (10-3-11) and (10-3-12). Since  $\tau$  relates to  $(x, t)$  by  $\tau = (Vt - z)/\rho$ , (10-3-27) can be used to solved  $k_z$  as a function of  $\tau$ . However, since this equation is very complicated, this can not be done analytically. Note that even for the simple case of  $\omega = \text{constant}$ , expressing  $k_z$  in terms of  $\tau$  explicitly is till very difficult, if not impossible. So, the points of stationary phase for a given  $\tau$  must be found numerically by using (10-3-27). This can be done without much difficulty if we can approximately find the points of stationary phase with the help of the graph of DWS and then search for the actual points of stationary phase near the approximate values by using (10-3-27). (10-3-26) also has to be used to determine the normal directions  $\hat{N}$  at the points of stationary phase. Thus, we see that the graph of DWS is very useful in the determination of the points of stationary phase.

The Gaussian curvature can be calculated by direct differentiation of (10-3-27) and then substitute into (3-3-8) - (3-3-10). Rewrite (10-3-27) by using  $\mathcal{D} = 0$ :

$$\tau = [2k_\rho^2(\Lambda' + \dot{\Lambda}c\beta) - (\Lambda' + \dot{\Lambda}c\beta)] / 4k_\rho(k_\rho^2 - \Lambda) \quad (10-3-28)$$

$$\frac{d^2 k_\rho}{dk_z^2} = \frac{d\tau}{dk_z} = \frac{1}{4k_\rho(k_\rho^2 - \Lambda)} \left\{ 4(\Lambda - 3k_\rho^2)\tau^2 + 8k_\rho\tau(\Lambda' + \dot{\Lambda}c\beta) \right\}$$

$$+2k_{\rho}^2[\Lambda'' + 2\dot{\Lambda}'c\beta + \dot{\Lambda}'(c\beta)^2] - [\Delta'' + 2\dot{\Delta}'c\beta + \dot{\Delta}'(c\beta)^2]\} \quad (10-3-29)$$

where

$$\Lambda'' = -(\lambda_2 + 1) \quad (10-3-30)$$

$$\Delta'' = 4\lambda_2[3k_z^2 - (\frac{\omega}{c})^2 S] \quad (10-3-31)$$

$$\dot{\Lambda}' = -\dot{\lambda}_2 k_z \quad (10-3-32)$$

$$\dot{\Delta}' = \dot{\Delta}' \dot{\lambda}_2 / \lambda_2 - 4\lambda_2[\frac{2\omega}{c^2} S + (\frac{\omega}{c})^2 \dot{S}] \quad (10-3-33)$$

$$\dot{\Lambda}'' = \frac{\lambda_1}{c^2} + \frac{2\omega}{c^2} \dot{\lambda}_1 + \frac{1}{2}(\frac{\omega}{c})^2 \dot{\lambda}_1'' - \frac{1}{2} \dot{\lambda}_2'' k_z^2 \quad (10-3-34)$$

$$\begin{aligned} \dot{\Delta}'' = & \frac{2\dot{\Delta}\dot{\lambda}_2}{\lambda_2} - \frac{2\Delta\dot{\lambda}_2^2}{\lambda_2^2} + \frac{\Delta\dot{\lambda}_2''}{\lambda_2} + \frac{\lambda_2}{c^4} \left\{ 2\omega^2\lambda_3 + 8\omega^3 \dot{\lambda}_3 + \omega^4\dot{\lambda}_3'' \right. \\ & \left. - 2(2S + 4\omega\dot{S} + \omega^2\dot{S}'')c^2k_z^2 \right\} \end{aligned} \quad (10-3-35)$$

$$\dot{\lambda}_1'' = \dot{P}'' + \frac{\dot{\lambda}_3''}{S} - \frac{2\dot{\lambda}_3}{S^2} \dot{S} + \frac{2\lambda_3\dot{S}^2}{S^3} - \frac{\lambda_3\dot{S}''}{S^2} \quad (10-3-36)$$

$$\dot{\lambda}_2'' = \frac{\dot{P}''}{S} - \frac{2\dot{P}\dot{S}}{S^2} + \frac{2P\dot{S}^2}{S^3} - \frac{P\dot{S}''}{S^2} \quad (10-3-37)$$

$$\dot{\lambda}_3'' = R\dot{L}'' + 2\dot{R}\dot{L} + L\dot{R}'' \quad (10-3-38)$$

$$\begin{aligned} \dot{S}'' = & (S-1)\frac{\dot{Y}''}{Y} + [\dot{S} - (S-1)\frac{\dot{Y}}{Y}][\frac{\dot{X}}{X} + \frac{2\dot{Y}}{Y} + \frac{\dot{X}''}{X}] \\ & - 8[\frac{1}{(1-X^2)^3} + \frac{M}{(M^2-X^2)^3}]X^2 \dot{X}^2 Y \end{aligned} \quad (10-3-39)$$

$$\dot{P}'' = -(1 + 1/M)\dot{Y}''$$

$$\begin{aligned} \dot{R}'' = & (R-1)\frac{\dot{Y}''}{Y} + [\dot{R} - (R-1)\frac{\dot{Y}}{Y}][\frac{2\dot{Y}}{Y} + \frac{\dot{X}''}{X}] \\ & - 2[\frac{1}{(1-X)^3} + \frac{M}{(M+X)^3}]\dot{X}^2 Y \end{aligned} \quad (10-3-40)$$

$$\begin{aligned} \dot{L}'' = & (L-1)\frac{\dot{Y}''}{Y} + [\dot{L} - (L-1)\frac{\dot{Y}}{Y}][\frac{2\dot{Y}}{Y} + \frac{\dot{X}''}{X}] \\ & - 2[\frac{1}{(1+X)^3} + \frac{M}{(M-X)^3}]\dot{X}^2 Y \end{aligned} \quad (10-3-40)$$



$$\dot{X}^* = -2\Omega_e / \omega^3 = -2X / \omega^2 \quad (10-3-41)$$

$$\dot{Y}^* = 6\omega_{pe}^2 / \omega^4 = 6Y / \omega^2 \quad (10-3-42)$$

So, we see that  $k_\rho''$  can be found by straightforward calculation without much difficulty, although quite lengthy.

Then, through

$$\lambda_{kz} = k_\rho'' / (1 + \tau^2)$$

$$\lambda_\phi = -1 / k_\rho (1 + \tau^2)$$

$$K = |k_\rho''| / k_\rho (1 + \tau^2)^2$$

the Gaussian curvature can be found in principle if a point of stationary phase is given.

For a given current density  $J(k)$ , putting (10-1-17), (10-3-8), (10-3-14), (10-3-27) into (2-20), the radiation field can then be found.

One special case that makes use of the above equations is the radiation from the point of stationary phase where  $\lambda_\phi = 0$ . Enhanced radiation field is found along the  $\hat{z}$  - axis. The discussion will be given in chapter XI.

## CHAPTER XI

### ENHANCED FIELD-ALIGNED RADIATION IN A COLD MAGNETOPLASMA

#### 11.1 Condition for $\lambda_\phi = 0$

From § 10-1, we know that,  $\lambda_\phi = 0$  may appear even for an ordinary wave-vector surface ( $\beta = 0$ ). We now find the condition for  $\lambda_\phi = 0$  for this case first.

Note that  $\lambda_\phi = 0$  appear when  $dk_z/dk_\rho = 0$  for  $k_\rho \neq 0$ , using (3-2-24) :

$$\frac{dk_z}{dk_\rho} = -(1 + \frac{S}{P}) (k_\rho/2k_z) \pm \left[ (1 - \frac{S}{P})^2 c^2 k_\rho^3 - 2\omega^2 D^2 \frac{k_\rho}{P} \right] / 4k_z \sqrt{\frac{1}{4}(1 - \frac{S}{P})^2 c^4 k_\rho^4 + \omega^2 D^2 (\omega^2 - \frac{c^2 k_\rho^2}{P})} \quad (11-1-1)$$

Set it equal to zero, it can be found that for  $k_\rho \neq 0$  :

$$c^2 k_\rho^2 = (\frac{\omega}{S-P})^2 \left\{ 2PD^2 \pm (\frac{S+P}{S}) D \sqrt{SP(P-R)(L-P)} \right\} \quad (11-1-2)$$

Substitute this back into (3-2-18), we find that :

$$c^2 k_z^2 = (\frac{\omega}{S-P})^2 \left\{ SP(P-R)(P-L) - PD^2 \pm \sqrt{SP(P-R)(L-P)} \right\} \quad (11-1-3)$$

The same equation can be found by putting the square-root term of (3-2-25) equal to zero. The reason is that in the case of  $k_z(k_\rho)$  having maximum or minimum when  $k_\rho \neq 0$ , the two branches (+, -sign) of (3-2-25) should coincide.

Note that (11-1-2) or (11-1-3) is the necessary condition. In order to see whether they really give the situation for  $\lambda_\phi = 0$ , the value of  $k_\rho$  or  $k_z$  should be put back into (3-2-24) or (3-2-25) to see whether the wave-vector surface exists in such cases.

It is interesting to find out in which region of the X,Y



parameter space (CMA diagram) that the  $\lambda_\phi = 0$  case must exist. In order to do so, we make use of the following fact :

(1) There are only three kinds of topology for the wave-vector surface in cold magnetoplasma :

(i) Surface exists for all  $\theta$  from 0 to  $\pi/2$  (spheroid), see Fig.11-1a.

(ii) Surface exists only for  $\theta$  from 0 to a resonance angle  $\theta_r$  (hyperboloid of two sheets), see Fig.11-1b.

(iii) surface exists only for  $\theta$  from  $\theta_r$  to  $\pi/2$  (hyperboloid of one sheet), see Fig.11-1c.

(2) On a fixed  $\phi$  plane, for every  $k_z$ , there are at most two  $k_\rho$  values.

Using these two facts, the existence of  $\lambda_\phi = 0$  can be discussed for each kinds of topology of the wave-vector surface :

(i) Spheroid :

If  $dk_z/dk_\rho > 0$  near  $k_\rho = 0$ , i.e. the surface initially goes upward at  $k_\rho = 0$ , there must be  $\lambda_\phi = 0$  case. The reason is that the surface must go down to the  $k_\rho$  axis and so a maximum of  $k_z$  must exist for a certain  $k_\rho \neq 0$ . On the other hand, if  $dk_z/dk_\rho < 0$  near  $k_\rho = 0$ , there will be  $\lambda_\phi \neq 0$  always, otherwise there will be more than two  $k_\rho$  values corresponding to a single  $k_z$  (see Fig. 11-2a).

(ii) Hyperboloid of two sheets :

Similar consideration leads to that if  $dk_z/dk_\rho > 0$  near  $k_\rho = 0$  there is no  $\lambda_\phi = 0$  case ; if  $dk_z/dk_\rho < 0$  near  $k_\rho = 0$ , this case must exist.

(iii) Hyperboloid of one sheet :

We have  $\lambda_\phi \neq 0$  always, otherwise this will also bring more than two  $k_\rho$  values to one  $k_z$  (see Fig 11-2b).



Note also that for (i) and (ii),  $\lambda_\phi = 0$  cannot take place at more than one point on the same surface (at a fixed  $\phi$ ). From the above consideration we should now calculate  $dk_z/dk_\rho$  at  $k_\rho \rightarrow 0$ . First, we note that by (3-2-24) :

$$n_{zo}^2 \equiv (ck_z/\omega)^2|_{k_\rho=0} = S \pm |D| \quad (11-1-4)$$

With (11-1-1), we found that, in the limit of  $k_\rho \rightarrow 0$ ,

$$\frac{dk_z}{dk_\rho} \rightarrow -\frac{k_\rho}{k_z} \left(1 + \frac{n_{zo}^2}{P}\right) \quad (11-1-5)$$

Then the condition for the existence of  $\lambda_\phi = 0$  are as follows:

(i) Spheroid :

The condition is  $1 + n_{zo}^2/P < 0$ . We immediately see that the above condition can never be satisfied for  $P > 0$ . Note that  $P \cong 1 - Y$  with  $Y \equiv (\omega_{pe}/\omega)^2$ . For  $P < 0$  and for  $n_{zo}^2 = R$ , i.e. the R-mode, the condition becomes :  $R > -P$ , and for L-mode, the condition is  $L > -P$ .

(ii) Hyperboloid of two sheets

The condition is  $1 + n_{zo}^2/P > 0$ . For  $P > 0$ , this condition is always satisfied. However,  $P > 0$  does not allow a hyperboloid with two sheets (see Fig.10-7). For  $P < 0$ , R-mode's condition is  $R < -P$ ; and for L-mode, it is  $L < -P$ .

So, if we plot the curves  $R = -P$  and  $L = -P$  on the CMA diagram (see Figs 10-4 and 10-7), these curves will separate the regions with or without the existence of  $\lambda_\phi = 0$  case. By the above condition, the region that  $\lambda_\phi = 0$  can happen has been indicated out at Fig 10-7, and is summarized as follow :

- (a) For regions inside  $P > 0$  (i.e.  $Y < 1$ ),  $\lambda_\phi$  is always non-zero everywhere on the wave-vector.
- (b) For the spheroid case, the  $\lambda_\phi = 0$  case exist in regions 4(a), 7(a), 11(a) and 12(a) (see Fig.10-7).



(c) For the Hyperboloid with two sheets case, the  $\lambda_\phi = 0$  case exist in regions 7(c), 8(b), 13(c).

Note that if both spheroid and Hyperboloid with two sheets exist in a parameter region, they cannot both have  $\lambda_\phi = 0$ , since  $n_{z0}$  for the two-sheeted hyperboloid case must be larger than for spheroid case. If  $\lambda_\phi = 0$  exists on both surface, we have, by (11-1-5),

$$S - |D| > S + |D| \quad (11-1-6)$$

which is impossible. This can be seen also by the fact that the two curves  $L = -P$  and  $R = -P$  do not intersect in the CMA diagram except for  $X = 0$  or  $Y = 0$ .

A direct check can be given by putting  $P + R = 0$  or  $P + L = 0$  into (11-1-2). We could found that  $k_\rho = 0$ , i.e. the  $\lambda_\phi = 0$  case happens at the  $k_\rho = 0$  point. Also by (11-1-5) we see that  $dk_z/dk_\rho \rightarrow 0$  near  $k_\rho = 0$  in this case. This situation is drawn in Fig.11-3.

For DWS with  $\beta \neq 0$ , the condition for the existence of  $\lambda_\phi = 0$  is much more complicated. Although (11-1-2) and (11-1-3) are still valid, they are now complicated equations of  $k_z$  since  $S, P, D, R, L$  depend on  $k_z$  through  $\omega = (\omega_0/\gamma) + ck_z\beta$ . So, a explicit condition cannot be easily found. However, the  $k_z$  values for  $\lambda_\phi = 0$  can be calculated by numerically if we know the approximate value of them from the graphs of DWS. Some cases in Figs 10-8 and 10-9 have actually been calculated and the values of  $n_\rho$  and  $n_z$  ( $n \equiv ck/\omega_0$ ) are shown on some graphs (see e.g. Fig 10-8c). From the graphs, we see that the numerical values found are consistent with the approximate values seen from the scale of the graphs.

## 11.2 Formula for finding the radiation field



For  $\lambda_\phi = 0$ , i.e.  $K = 0$  also, (2-20) cannot be used to calculate the radiation field. Following the consideration of § 6-3, the far field is given by an equations similar to (6-3-9) :

$$E = \frac{e}{\pi} \frac{-i\omega_\sigma t/\gamma}{\sum \int_0^{2\pi} \left| \frac{\partial \mathbf{k}^\sigma}{\partial \phi} \right| d\phi \left\{ \left( \frac{2\pi}{|\lambda_{kz}^\sigma| |x - vt|} \right)^{\frac{1}{2}} \left[ \frac{\omega \bar{\Gamma}}{c \frac{\partial \mathcal{D}}{\partial \mathbf{k}} + \frac{\partial \mathcal{D}}{\partial \omega} \mathbf{v}} \right]_D \cdot \mathbf{J}(\mathbf{k}^\sigma) \exp\left(\frac{i\pi}{4} \text{sign}(\lambda_{kz}^\sigma) + i\mathbf{k}^\sigma \cdot (\mathbf{x} - \mathbf{vt})\right) \right\} \quad (11-2-1)$$

since  $dk_z/dk_\rho = 0$  at  $\lambda_\phi = 0$ , so  $\tau = \pm \omega$ . This means that the observation direction  $\mathbf{x} - \mathbf{vt}$  must be in the  $\hat{z}$  direction, so,

$$\mathbf{k} \cdot (\mathbf{x} - \mathbf{vt}) = k_z (\mathbf{x} - \mathbf{vt})_N \quad (11-2-2)$$

is independent of  $\phi$ . Also,  $|\partial \mathbf{k} / \partial \phi|$ ,  $\lambda_{kz}$  are given by (3-3-3) and (3-3-8) which obviously independent of  $\phi$ . Furthermore,  $\frac{\partial \mathcal{D}}{\partial \mathbf{k}}$  and  $\frac{\partial \mathcal{D}}{\partial \omega}$ , as given by (10-3-13) and (10-3-14), are also not functions of  $\phi$ . So, (11-2-1) becomes :

$$E = \sum \left( \frac{2}{\pi |\lambda_{kz}^\sigma| |z - vt|} \right)^{\frac{1}{2}} k_\rho \exp\left(\frac{i\pi}{4} \text{sign}(\lambda_{kz}^\sigma) + i\mathbf{k}_z^\sigma z - \omega^\sigma t\right) \left[ \frac{\omega^\sigma}{c \frac{\partial \mathcal{D}}{\partial \mathbf{k}} + \frac{\partial \mathcal{D}}{\partial \omega} \mathbf{v}} \right]_D \int_0^{2\pi} d\phi [\bar{\Gamma}(\mathbf{k}^\sigma, \omega^\sigma)]_D \cdot \mathbf{J}(\mathbf{k}^\sigma) \quad (11-2-3)$$

The magnetic field can be found similarly :

$$B = \sum \left( \frac{2}{\pi |\lambda_{kz}^\sigma| |z - vt|} \right)^{\frac{1}{2}} c k_\rho \frac{\exp\left(\frac{i\pi}{4} \text{sign}(\lambda_{kz}^\sigma) + i\mathbf{k}_z^\sigma z - \omega^\sigma t\right)}{\left[ c \frac{\partial \mathcal{D}}{\partial \mathbf{k}} + \frac{\partial \mathcal{D}}{\partial \omega} \mathbf{v} \right]_D} \int_0^{2\pi} d\phi \mathbf{k}^\sigma \times \left( [\bar{\Gamma}(\mathbf{k}^\sigma, \omega^\sigma)]_D \cdot \mathbf{J}(\mathbf{k}^\sigma) \right) \quad (11-2-4)$$

The time-averaged Poynting vector can then be calculated by :

$$\langle S \rangle = \frac{c}{16\pi} (\mathbf{E}^* \times \mathbf{B} + \mathbf{E} \times \mathbf{B}^*) \quad (11-2-5)$$

For a stationary source, i.e.  $\beta = 0$ , we have shown that there is at most one  $\lambda_\phi = 0$  point on the DWS. Since the field



given by (11-2-3) and (11-2-4) varies as  $|x - vt|^{-1/2}$  which dominate the far field unless the  $\phi$ -integration vanishes. So, we only need to consider one  $\alpha$  in the above equations, and therefore there is no mixed-mode interference in  $\langle S \rangle$ .

For  $\beta \neq 0$ , more than one  $\lambda_\phi = 0$  case may happen. However, since  $\omega$  depends on  $k_z$  now, so these stationary will have different frequencies. The interference of these  $k$ 's in the time-averaged Poynting will then average out. So, we can consider term by term in the above equations. Then we have,

$$\langle S \rangle = \frac{c^2 \omega k_\rho^2}{8\pi^2 |\lambda_{kz}| |z - vt| \left[ c \frac{\partial \mathcal{D}}{\partial k} + \frac{\partial \mathcal{D}}{\partial \omega} \mathbf{V} \right]_{N,D}^2} \cdot \left\{ \left[ \int_0^{2\pi} d\phi \bar{\mathbf{F}} \cdot \mathbf{J} \right]^* \times \left[ \int_0^{2\pi} d\phi \mathbf{k} \times (\bar{\mathbf{F}} \cdot \mathbf{J}) \right] + c.c. \right\} \quad (11-2-6)$$

vector

The  $|z - vt|^{-1}$  dependence of the Poynting  $\mathbf{A}$  is quite different from that of the ordinary case, however, the above result only valid on the  $z$ -axis and so it is called the "enhanced field-aligned radiation". For the observation direction  $x - vt$  not parallel to  $\hat{z}$ , the Poynting vector depends on  $|x - vt|^{-2}$  in general, so the energy conservation is not violated by this enhanced radiation. Also, the power received per unit solid angle at  $\theta = 0$  in this case cannot be found by the following equation :

$$dP/d\Omega = |\langle S \rangle| |x - vt'| = |\langle S \rangle| |x - vt|^2 V_E / |V_E - V|^2 \quad (11-2-7)$$

The actual calculation of this quantity may be complicated, however, from the above discussion, we see that the power radiated along  $\hat{z}$  must be very large compared with that along another direction.

### 11.3 Group velocity and $\lambda_{kz}$

The DWS is given by (10-3-26) :

$$k_{\rho}^2 = \xi_{\pm} \quad (11-3-1)$$

where  $\xi_{\pm}$  is given by (10-3-1). At the point where  $\lambda_{\phi} = 0$  :

$$\Theta = 0 \quad (11-3-2)$$

$$k_{\rho}^2 = \Lambda \quad (11-3-3)$$

$$\Lambda^2 = \Delta \quad (11-3-4)$$

By (10-3-9), (10-3-10) and (10-3-14) :

$$\partial D / \partial k_{\rho} = 0 \quad (11-3-5)$$

$$\partial D / \partial k_z = -c^4 \omega^2 \dot{\Theta}' \quad (11-3-6)$$

$$\partial D / \partial \omega = -c^4 \omega^2 \dot{\Theta} \quad (11-3-7)$$

So,

$$\mathbf{V}_E = - \frac{\partial D}{\partial \mathbf{k}} / \frac{\partial D}{\partial \omega} = - \frac{\dot{\Theta}'}{\dot{\Theta}} \hat{\mathbf{z}} \quad (11-3-8)$$

$$\mathbf{V}_E - \mathbf{V} = - \left( \frac{\dot{\Theta}'}{\dot{\Theta}} + c\beta \right) \hat{\mathbf{z}} \quad (11-3-9)$$

$$\left( \frac{\partial D}{\partial \mathbf{k}} + \frac{\partial D}{\partial \omega} \mathbf{V} \right)_N = - \frac{\partial D}{\partial \omega} |\mathbf{V}_E - \mathbf{V}| = c^4 \omega^2 \dot{\Theta} \left| \frac{\dot{\Theta}'}{\dot{\Theta}} + c\beta \right| \quad (11-3-10)$$

When  $\Theta = 0$ , (10-3-27) and (10-3-29) become (keeping the largest term in  $\Theta \rightarrow 0$ ) :

$$\tau = (\dot{\Theta}' + \dot{\Theta} c\beta) / 4k_{\rho} (k_{\rho}^2 - \Lambda) \quad (11-3-11)$$

$$k_{\rho}'' = (\dot{\Theta}' + \dot{\Theta} c\beta)^2 / 8k_{\rho} (k_{\rho}^2 - \Lambda)^3 \quad (11-3-12)$$

Substitute into (3-3-8) :

$$\lambda_{kz} = \pm 8k_{\rho}^2 / (\dot{\Theta}' + \dot{\Theta} c\beta) \quad (11-3-13)$$

Let

$$s_1 \equiv \text{sign}(\dot{\Theta}), \quad s_2 \equiv \text{sign}(\lambda_{kz}) \quad (11-3-14)$$

Then, (11-2-3), (11-2-5) and (11-2-6) can be rewritten as :

$$\mathbf{E} = \left[ \frac{1}{\pi |z - Vt|} \right]^{\frac{1}{2}} \frac{e^{i\pi s_2/4} e^{i(k_z z - \omega t)}}{2c^4 \omega s_1 |\dot{\Theta}' + \dot{\Theta} V|^{1/2}} \int_0^{2\pi} d\phi \bar{\mathbf{r}} \cdot \mathbf{J} \quad (11-3-15)$$

$$\mathbf{B} = \left[ \frac{1}{\pi |z - Vt|} \right]^{\frac{1}{2}} \frac{e^{i\pi s_2/4} e^{i(k_z z - \omega t)}}{2c^3 \omega^2 s_1 |\dot{\Theta}' + \dot{\Theta} V|^{1/2}} \int_0^{2\pi} d\phi \mathbf{k} \times (\bar{\mathbf{r}} \cdot \mathbf{J}) \quad (11-3-16)$$



$$\langle S \rangle = \frac{[\int d\phi \bar{\Gamma} \cdot J]^* \times [\int d\phi k_x (\bar{\Gamma} \cdot J)] + c.c.}{64\pi^2 c^6 \omega^3 |z - vt| |\dot{\Theta}' + \dot{\Theta}V|} \quad (11-3-17)$$

Note that  $\dot{\Theta}'$  and  $\dot{\Theta}$  can be found easily by (10-3-2) and (10-2-15) - (10-3-25) provided that a point of stationary phase  $k_z$ , where  $\lambda_\phi(k_z) = 0$ , is given. For  $\beta = 0$ , the calculation is much more simple since we do not have to calculate  $\dot{\Theta}$ .

#### 11.4 Integration over $\phi$

The integration over  $\phi$  in the above equations depends on the choice of the current density. The most natural radiating source moving along the field of a magnetoplasma is the helically moving charge described in § 4.2.1. The  $J(k, \omega)$  is given by (4-2-13) and (4-2-14). By (10-1-3) and (10-1-4) :

$$\hat{x} + i\hat{y} = e^{i\phi} (\hat{\rho} + i\hat{\phi}) \quad (11-4-1)$$

$$\hat{x} - i\hat{y} = e^{-i\phi} (\hat{\rho} - i\hat{\phi}) \quad (11-4-2)$$

So, (4-2-14) can be rewritten as :

$$J_n(k) = q \left( \frac{v_\perp}{2\gamma} [(J_{n+1} + J_{n-1})\hat{\rho} + (J_{n+1} - J_{n-1})\hat{\phi}] + vJ_n \hat{z} \right) \exp -in\phi \quad (11-4-3)$$

Note that the argument of the Bessel functions is  $k_\rho v_\perp / \Omega$ , so they are independent of  $\phi$ . Define :

$$J_n(k) \equiv [J_{n\rho} \hat{\rho} + J_{n\phi} \hat{\phi} + J_{nz} \hat{z}] \quad (11-4-4)$$

where  $J_{n\rho}$ ,  $J_{n\phi}$  and  $J_{nz}$  are independent of  $\phi$ .

Using the  $\hat{\rho}$ ,  $\hat{\phi}$ ,  $\hat{z}$  representation of  $\bar{\Gamma}$ , we have

$$\bar{\Gamma} \cdot J_n = e^{-in\phi} [G_{n\rho} \hat{\rho} + G_{n\phi} \hat{\phi} + G_{nz} \hat{z}] \quad (11-4-5)$$

where

$$G_{n\rho} = \Gamma_{\rho\rho} J_{n\rho} + \Gamma_{\rho\phi} J_{n\phi} + \Gamma_{\rho z} J_{nz} \quad (11-4-6)$$

$$G_{n\phi} = \Gamma_{\phi\rho} J_{n\rho} + \Gamma_{\phi\phi} J_{n\phi} + \Gamma_{\phi z} J_{nz} \quad (11-4-7)$$

$$G_{nz} = \Gamma_{z\rho} J_{n\rho} + \Gamma_{z\phi} J_{n\phi} + \Gamma_{zz} J_{nz} \quad (11-4-8)$$

which are also not functions of  $\phi$ . Using the fact that :

$$\int_0^{2\pi} e^{-in\phi} \cos\phi d\phi = \pi(\delta_{n,1} + \delta_{n,-1}) \quad (11-4-9)$$

$$\int_0^{2\pi} e^{-in\phi} \sin\phi d\phi = -i\pi(\delta_{n,1} - \delta_{n,-1}) \quad (11-4-10)$$

we have,

$$\begin{aligned} \int_0^{2\pi} e^{-in\phi} \hat{\rho} d\phi &= \pi[(\delta_{n,1} + \delta_{n,-1})\hat{x} - i(\delta_{n,1} - \delta_{n,-1})\hat{y}] \\ \int_0^{2\pi} e^{-in\phi} \hat{\phi} d\phi &= \pi[i(\delta_{n,1} - \delta_{n,-1})\hat{x} + (\delta_{n,1} + \delta_{n,-1})\hat{y}] \end{aligned} \quad (11-4-11)$$

Therefore, only the  $n = 0, \pm 1$  terms can survive after the integration of  $\int \bar{\Gamma} \cdot J d\phi$ . By the argument in § 4-3, the  $n = \pm 1$  terms can be combined into just the  $n = 1$  term multiplied by two and take the real part of the final result. For  $n = 1$ :

$$\begin{aligned} E_{n=1} &\propto \int_0^{2\pi} d\phi \bar{\Gamma} \cdot J_1 \times 2 \\ &= 2\pi(G_\rho + iG_\phi)(\hat{x} - i\hat{y}) \end{aligned} \quad (11-4-12)$$

By a similar consideration :

$$\begin{aligned} B_{n=1} &\propto \int_0^{2\pi} d\phi \mathbf{k} \times (\bar{\Gamma} \cdot J_1) \times 2 \\ &= 2\pi[-k_z G_\phi + i(k_z G_\rho - k_\rho G_z)](\hat{x} - i\hat{y}) \end{aligned} \quad (11-4-13)$$

where now the  $n = 1$  index in  $G$  is omitted. Since  $E$  and  $B$  also depend on  $\exp(i(k_z z - \omega t))$ , so for  $\omega > 0$ , the wave is left hand circularly polarized if it propagate along the  $z$ -direction and is right hand circularly polarized if it propagates along the  $-z$ -direction. This polarization is the same as that of the circularly moving charge. Note also that

$E_1 \perp B_1$ , since

$$(\hat{x} - i\hat{y}) \cdot (\hat{x} - i\hat{y}) = 0 \quad (11-4-14)$$



and the direction of Poynting vector is along  $\pm \hat{z}$  :

$$(\hat{x} - i\hat{y})^* \times (\hat{x} - i\hat{y}) = -2i\hat{z} \quad (11-4-15)$$

For the  $n = 0$  term,

$$E_0 \propto G_{0z} \hat{z} \quad (11-4-16)$$

$$B_0 \propto k_{\rho} G_{0\rho} \hat{z} \quad (11-4-17)$$

So,  $E_0 \times B_0 = 0$ , i.e. there is no radiation for the  $n = 0$  term. Note that this  $n = 0$  term corresponds to Cerenkov radiation, so we conclude that when  $\lambda_{\phi} = 0$ , there is no Cerenkov radiation.

We can now write down the total radiation field ( for one  $\sigma$  ) by using (11-4-12), (11-4-13) and (11-4-15) :

$$E = \left[ \frac{\pi}{|z - vt|} \right]^{\frac{1}{2}} \frac{e^{i\pi s_2/4} e^{i(k_z z - \omega t)}}{c^4 \omega s_1 |\Theta' + \dot{\Theta}V|^{1/2}} (G_{\rho} + iG_{\phi}) (\hat{x} - i\hat{y})$$

$$B = \left[ \frac{\pi}{|z - vt|} \right]^{\frac{1}{2}} \frac{e^{i\pi s_2/4} e^{i(k_z z - \omega t)}}{c^3 \omega^2 s_1 |\Theta' + \dot{\Theta}V|^{1/2}} [-k_z G_{\phi} + i(k_z G_{\rho} - k_{\rho} G_z)] (\hat{x} - i\hat{y})$$

$$\langle S \rangle = \frac{\text{Re} -i(G_{\rho}^* - iG_{\phi}^*) [-k_z G_{\phi} + i(k_z G_{\rho} - k_{\rho} G_z)]}{4 c^3 \omega^3 |z - vt| |\Theta' + \dot{\Theta}V|} \hat{z} \quad (11-4-18)$$

Note that  $G_{\rho}$ ,  $G_{\phi}$  and  $G_z$  are given by (11-4-6) - (11-4-8) and the component of  $\bar{\Gamma}$  in  $\hat{\rho}$ ,  $\hat{\phi}$ ,  $\hat{z}$  are given by (10-1-9). We can see that:

$$G_{\rho}^* = G_{\rho}, \quad G_z^* = G_z \quad (11-4-19)$$

$$G_{\phi}^* = -G_{\phi} \quad (11-4-20)$$

define :

$$\tilde{G}_{\phi} = -i G_{\phi} \quad (11-4-21)$$

Then (11-4-18) can be rewritten as :

$$\langle S \rangle = \frac{(G_{\rho} - \tilde{G}_{\phi}) [k_z (G_z - \tilde{G}_{\phi}) - k_{\rho} G_z]}{4 c^3 \omega^3 |z - vt| |\Theta' + \dot{\Theta}V|} \hat{z} \quad (11-4-22)$$

## 11.5 Self mode interference of the enhanced radiation

We now use the linearly oscillating charge as the moving radiating source to perform the  $\phi$ -integration. Assumed that it is moving along the z-axis and oscillating in the x-z plane. Also, we use the dipole approximation so that we only have to consider the simplest expression of the current density :

$$J(k) = \frac{q}{2} \left[ \frac{\omega_0 r_0}{\gamma} + (k_\rho x_0 \cos \phi + k_z \frac{z_0}{\gamma}) V \hat{z} \right] \quad (11-5-1)$$

where

$$r_0 = x_0 \hat{x} + (z_0 / \gamma) \hat{z} \quad (11-5-2)$$

with  $x_0$  and  $z_0$  being two constants. Since now this source is not cylindrical symmetric, it may be more convenient to use the  $\hat{x}$ ,  $\hat{y}$ ,  $\hat{z}$  representation of  $\vec{r}$ , i.e. (10-1-17). Then ,

$$\vec{r} \cdot J = \frac{q}{2} \left[ \begin{aligned} & \frac{\omega_0}{\gamma} x_0 (a_1 + a_2 \cos^2 \phi) + \left( \frac{\omega z_0}{\gamma} + k_\rho V x_0 \cos \phi \right) (a_5 \cos \phi + i a_4 \sin \phi) \\ & \frac{\omega_0}{\gamma} x_0 (a_2 \sin \phi - i a_3) + \left( \frac{\omega z_0}{\gamma} + k_\rho V x_0 \cos \phi \right) (a_5 \sin \phi + i a_4 \cos \phi) \\ & \frac{\omega_0}{\gamma} x_0 (a_5 \cos \phi + i a_4 \sin \phi) + \left( \frac{\omega z_0}{\gamma} + k_\rho V x_0 \cos \phi \right) a_6 \end{aligned} \right] \quad (11-5-4)$$

Note that:

$$\int_0^{2\pi} \sin \phi \, d\phi = \int_0^{2\pi} \cos \phi \, d\phi = \int_0^{2\pi} \sin \phi \cos \phi \, d\phi = 0 \quad (11-5-5)$$

$$\int_0^{2\pi} \sin^2 \phi \, d\phi = \pi \quad (11-5-6)$$

then ,

$$\int_0^{2\pi} \vec{r} \cdot J \, d\phi = \frac{\pi q}{2} \left[ \begin{aligned} & 2 \frac{\omega_0}{\gamma} x_0 a_1 + \frac{\omega_0}{\gamma} x_0 a_2 + k_\rho V x_0 a_5 \\ & -i 2 \frac{\omega_0}{\gamma} x_0 a_3 + i k_\rho V x_0 a_4 \\ & 2 \frac{\omega_0}{\gamma} z_0 a_6 \end{aligned} \right] \quad (11-5-7)$$

For the B field, we similarly get



$$\int_0^{2\pi} \mathbf{k} \times (\bar{\Gamma} \cdot \mathbf{J}) d\phi = \frac{\pi q}{2} .$$

$$\left[ \begin{array}{l} i \frac{\omega}{\gamma} k_{\rho} x_0 a_4 + i k_z x_0 [2 \frac{\omega}{\gamma} a_3 - k_{\rho} V a_4] \\ k_z x_0 [\frac{\omega}{\gamma} (2a_1 + a_2) + k_{\rho} V a_5] - k_{\rho} x_0 [\frac{\omega}{\gamma} a_5 + k_{\rho} V a_6] \\ 2i\omega k_{\rho} z_0 a_4 / \gamma \end{array} \right] \quad (11-5-8)$$

By straightforward calculation, we arrive at,

$$\left[ \int_0^{2\pi} \bar{\Gamma} \cdot \mathbf{J} d\phi \right]^* \times \left[ \int_0^{2\pi} \mathbf{k} \times (\bar{\Gamma} \cdot \mathbf{J}) d\phi \right] + \text{c.c.} \\ = \left( \frac{\pi q}{2} \right)^2 \left[ \begin{array}{l} -4x_0 z_0 \omega [(\omega_0 / \gamma) \eta_1 + k_{\rho} V \eta_2] / \gamma \\ 0 \\ 2x_0^2 [(\frac{\omega}{\gamma})^2 \eta_3 + (\frac{\omega}{\gamma}) k_{\rho} V \eta_4 + k_{\rho}^2 V^2 \eta_5] \end{array} \right] \quad (11-5-9)$$

where

$$\eta_1 = k_{\rho} (2a_3 a_4 - a_5 a_6) + a_6 (2a_1 + a_2) k_z \quad (11-5-10)$$

$$\eta_2 = k_{\rho} (a_4^2 - a_6^2) + k_z a_5 a_6 \quad (11-5-11)$$

$$\eta_3 = (2a_1 + a_2) [(2a_1 + a_2) k_z - a_5 k_{\rho}] - 2a_3 (k_{\rho} a_4 + 2a_3 k_z) \quad (11-5-12)$$

$$\eta_4 = (2a_1 + a_2) [2k_z a_5 - k_{\rho} a_6] - k_{\rho} (a_5^2 - a_4^2) + 4a_3 a_4 k_z \quad (11-5-13)$$

$$\eta_5 = k_z (a_5^2 - a_4^2) - k_{\rho} a_5 a_6 \quad (11-5-14)$$

We see that the Poynting vector is not in the z-direction. The x-component of the Poynting vector does not vanish in the general, except for  $z_0 = 0$ . (For  $x_0 = 0$ ,  $S = 0$  since the charge is oscillating along the observation direction). By this, it appears that the non-vanishing x-component of the time-average Poynting vector is due to the interference of the wave's of different  $\phi$ . If  $z_0 = 0$ , then the charge is oscillating in the perpendicular plane of the z-direction, so the source is symmetric in x and -x direction (note that since the charge is oscillating on the x-z plane, the source is also symmetric in the y-direction), thus, the

Poynting vector must point along  $z$ , i.e. no interference effect. If  $z_0 \neq 0$ , the source is not symmetric in  $x$  and  $-x$  direction, so interference effect happens. From this explanation, the fact that this interference effect does not exist in the helically moving charge is also expected since this current density is cylindrically symmetric.

Note that this interference effect is due to the same mode. The difference from this to the mixed mode interference is that this effect may also exist for the moving source while the mixed-mode interference effect is hard to appear for the case of moving source since the frequencies of the wave's emitted by the two modes are probably different from each other due to the Doppler effect.

For the  $\beta = 0$  case, this self-mode interference effect actually exists not only on the  $z$ -axis, but also in the region for  $\theta$  not larger than a certain value. This is because there are three points of stationary phase on the surface with the same normal direction (see Fig 11-4). So the interference effect of the enhanced field-aligned radiation is not surprising.



---

## CHAPTER XII

### CONCLUSIONS

In this thesis, the formulation given earlier by Lai and Chan has been applied in a systematic way to evaluate the far field and energy flow caused by a radiating source moving in a number of media.

(1) It has been shown that the point of stationary phase of a quadratic Doppler-shifted wave-vector surface (DWS), whether cylindrically symmetric or not, can be expressed in terms of observer's position and time explicitly. As a result the far field and energy flow has been evaluated in a straight forward way for several media, including a non-dispersive uniaxial medium, an isotropic cold plasma and an uniaxial cold magnetoplasma (in the strong field limit and for the charge moving along the magnetic field).

(2) For a more complicated media, the point of stationary phase for a given observation point has to be found by numerical method. The graph of DWS gives a lot of help in this numerical calculation. This graph can be plotted without much difficulty if the surface is cylindrically symmetric and  $k_{\perp}$  can be expressed as a function of  $\omega$  and  $k_z$ . Graphs of DWS of cold magnetoplasma for the radiating source moving along the magnetic field has actually been plotted and points of stationary phase have been found numerically for some special cases.

(3) For a charge moving along the magnetic field line in the uniaxial cold magnetoplasma, the DWS is an ellipsoid

because of dispersion and the Cerenkov far field calculated is quite different from that in a non-dispersive medium.

(4) For a radiating source (of proper frequency  $\omega_c$ ) moving in a cold plasma, we have found that the DWS moves up into the upper region of the wave-vector space if  $\omega_c > \gamma\omega_p$  and consequently, Frank's complex Doppler effect occurs and the radiation is all-forward.

(5) For a radiating source moving along the field in a cold magnetoplasma, we have shown that, depending on the velocity of the source, enhanced radiation flux (which is inversely proportional to distance) could appear along the field direction, forward or backward.

(6) From the above study, we see that many interesting effect may appear in the Doppler effect for radiating source moving in an anisotropic dispersive medium. It is natural to think that these special <sup>features</sup> have not yet been completely known. The concept of DWS may be very helpful in the study of these effects.



---

## REFERENCES

- Chen, H.C., *Theory of Electromagnetic Waves* (McGraw-Hill, New York, 1983).
- Frank, I.M., J. Phys. U.S.S.R., 7(2), 49 (1943).
- Giles, M.J., J. Plasma Phys. 19, 201 (1978).
- Ginzburg, V.L., Usp. Fiz. Nauk, 69, 537 (1959) [Sov. Phys. Usp. 2, 876 (1960).]
- Jackson, J.D., *Classical Electrodynamics* (Wiley, New York, 1975).
- Lai, H.M. and P.K. Chan, Phys. Fluids 29, 1881 (1986).
- Landau, L.D. and E.M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon New York, 1960).
- Lee, K.S.H. and C.H. Papas, IEEE trans. Anten & prop. 13(5), 799 (1965).
- Lighthill, M.J., Philos. Trans. R. Soc. (London) A252, 397 (1960).
- Papas, C.H., *Theory of Electromagnetic Waves Propagation* (McGraw-Hill, New York, 1965).
- Stix, T.H., *Theory of Plasma Waves* (McGraw-Hill, New York, 1962).

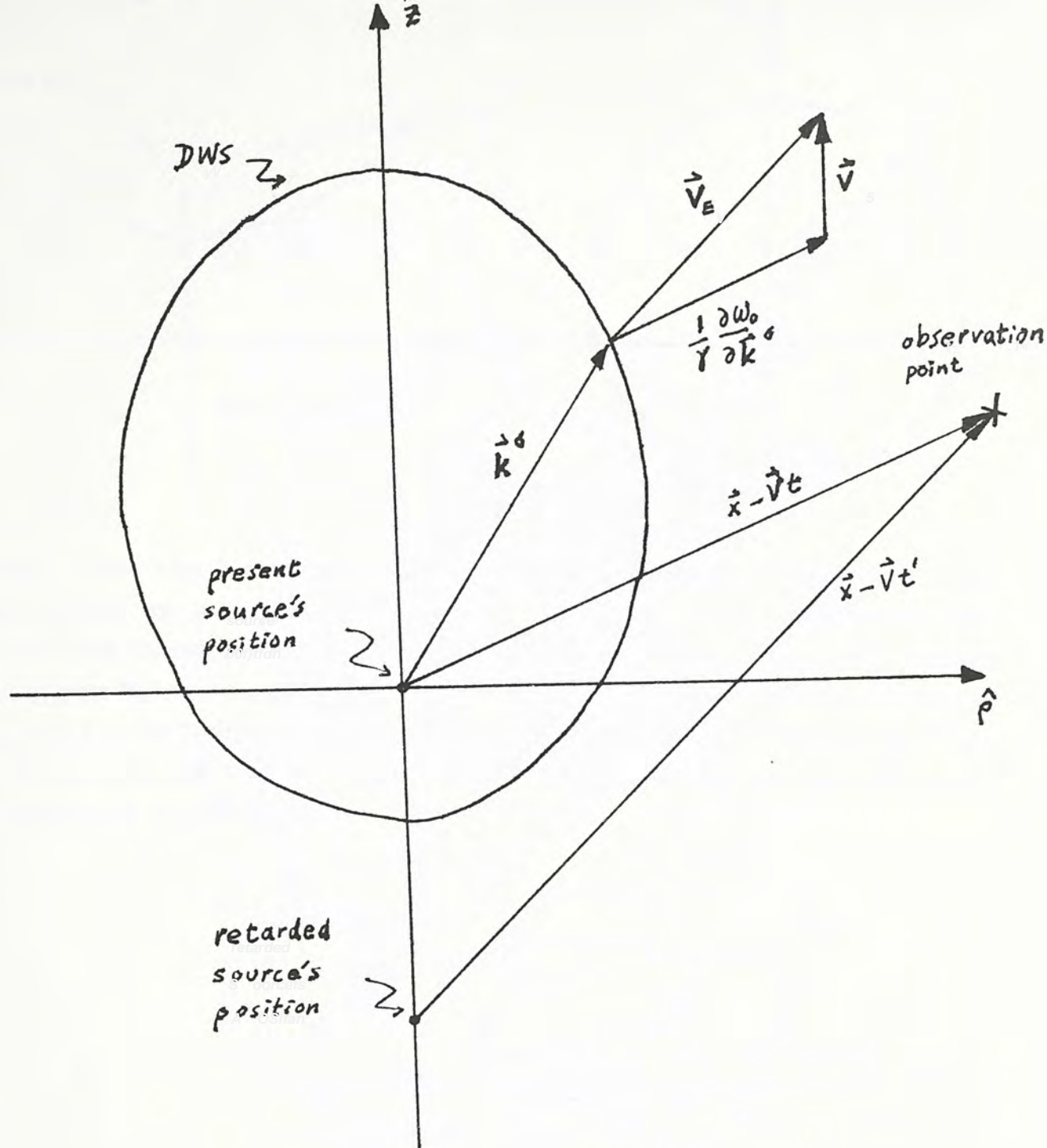
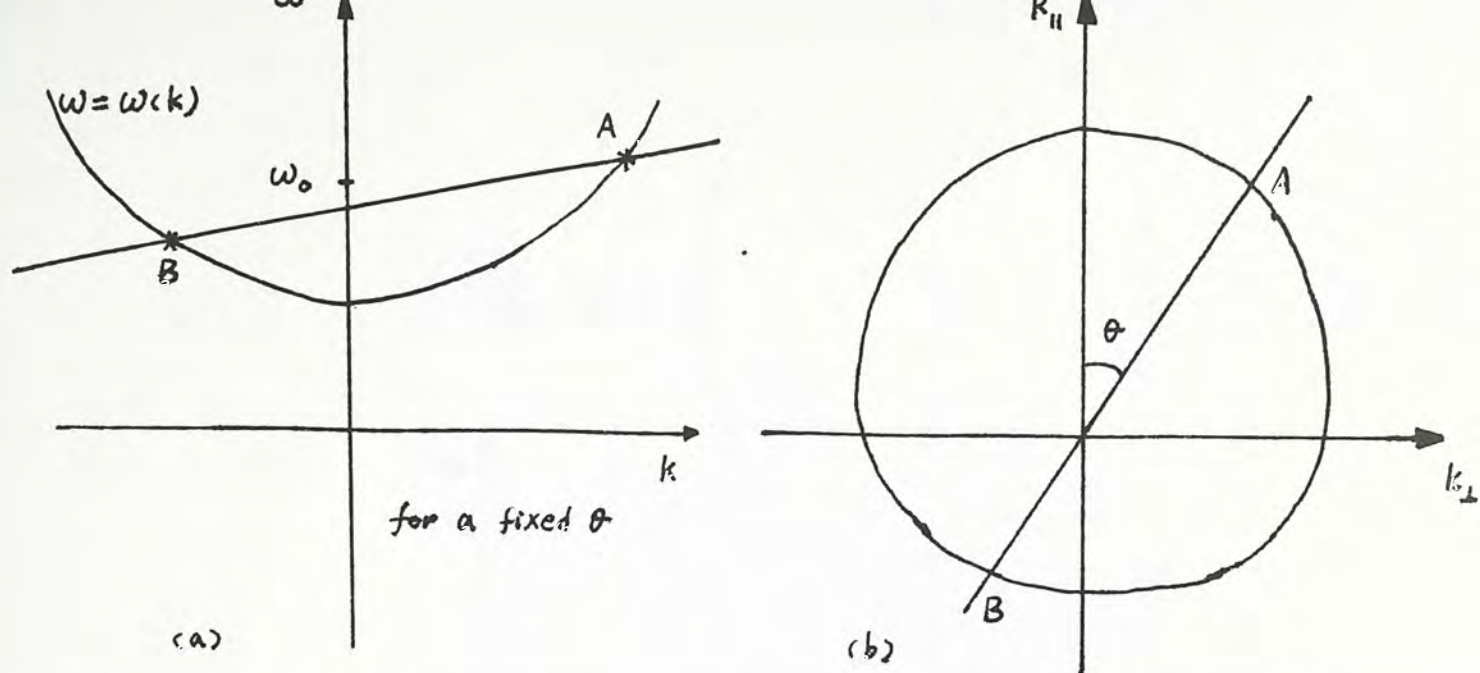
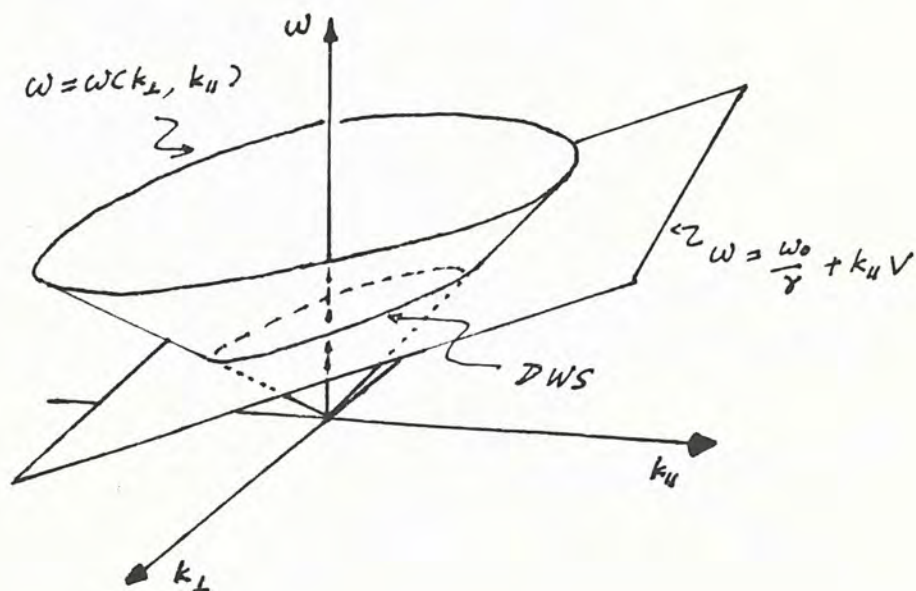


Fig. 2-1 The DWS for a moving radiating source, with velocity  $V = V \hat{z}$  and passing  $z = 0$  at  $t = 0$ , is drawn. For an observation point  $(x, t)$ , the point of stationary phase  $k^\sigma$  on the DWS is found by the condition that  $\partial \omega_0(k)/\partial k$  is parallel to  $x - Vt$ . The group velocity is found by  $V_E = \frac{1}{\gamma} \partial \omega_0(k)/\partial k + V$  and the retarded position of the source  $Vt'$  can also be found since  $x - Vt'$  is parallel to  $V_E$ .

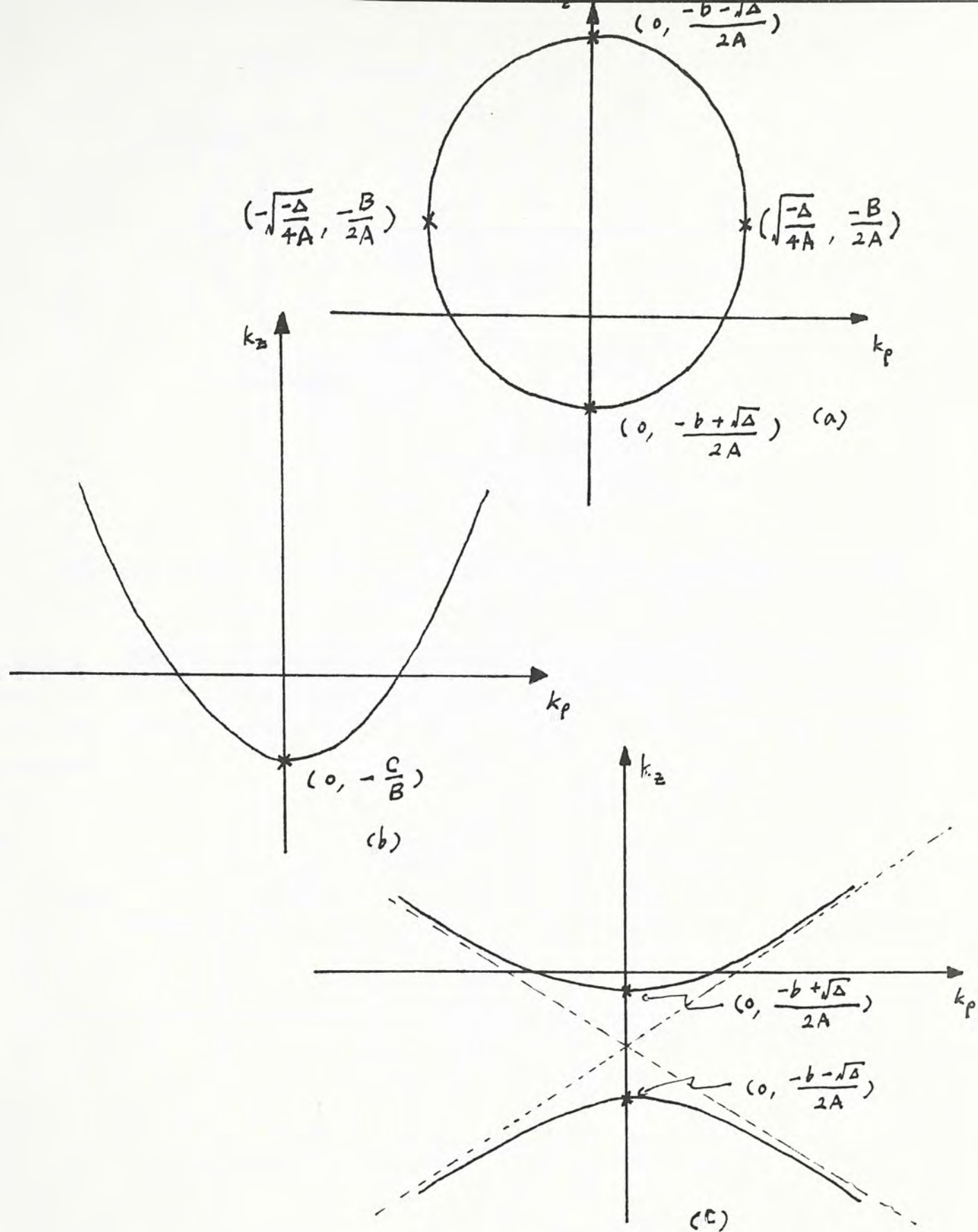




**Fig.3-1** For the isotropic case, the dispersion relation  $\omega = \omega(k)$  can be drawn as a curve in 2D space and this curve intersects the line of the Doppler equation  $\omega = (\omega_0/\gamma) + kV\cos\theta$  at some points for a given  $\theta$ , (a) ; The DWS is then can be drawn on the 2D space of  $k_\rho$  vs  $k_z$  as the curve of these intersection points vs  $\theta$ , (b). The actual DWS is the surface of rotation of this curve, so it is cylindrically symmetric.



**Fig.3-2** For an anisotropic medium with cylindrical symmetry about  $\hat{z}$ , the dispersion relation is of the form  $\omega = \omega(k_\rho, k_z)$ , so it can be drawn as a surface in a 3D  $\omega, k_\rho, k_z$  space. If the velocity  $V$  of the source is along  $\hat{z}$ , the the Doppler equation is a plane  $\omega = (\omega_0/\gamma) + k_z V$ . The intersection of this plane with  $\omega = \omega(k_\rho, k_z)$  is the DWS which is also cylindrically symmetric.



**Fig.3-3** DWS of the form  $k_p^2 = Ak_z^2 + Bk_z + C$ , with  $A, B, C$  being constants is a kind of quadratic surface having cylindrical symmetry about  $\hat{z}$ . There are three kinds of topologies : (a) for  $A < 0$ , it is an ellipsoid; (b) for  $A = 0$ , it is a paraboloid; (c) for  $A > 0$ , it is a hyperboloid.



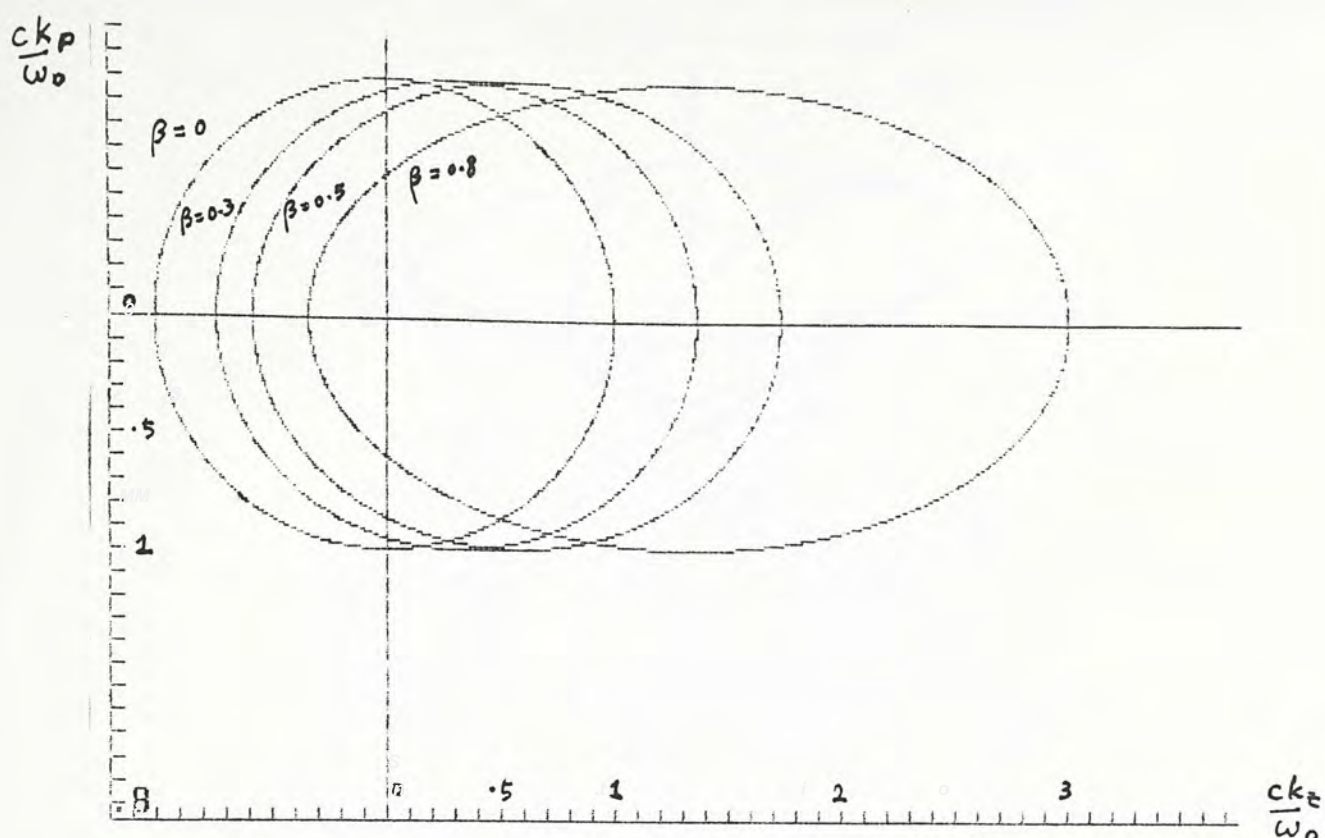


Fig. 3-4 DWS's of vacuum for different velocities are drawn. Note that for  $\beta \rightarrow 1$ , the length (along) of the DWS tends to infinity but its lowest point  $\rightarrow 0^-$ . The width of the DWS is always constant.

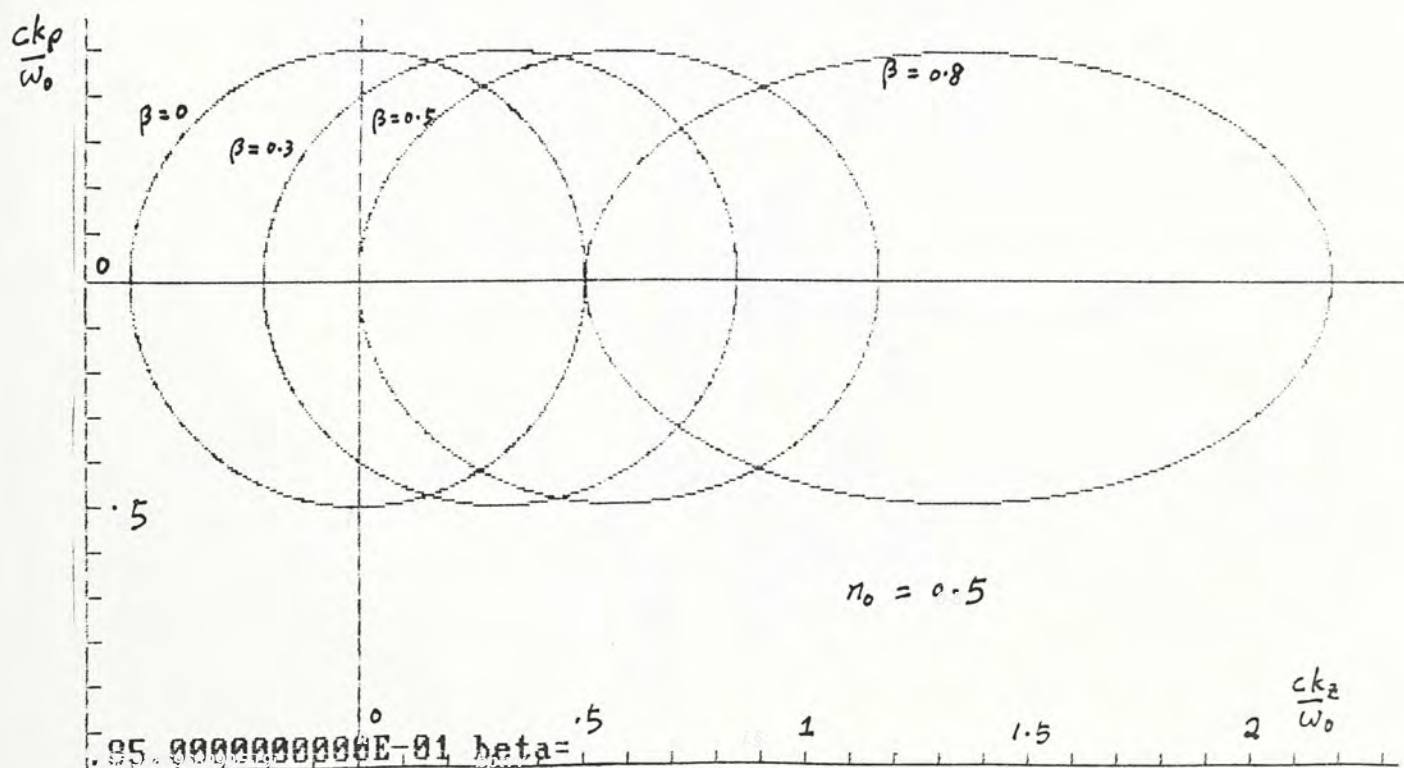
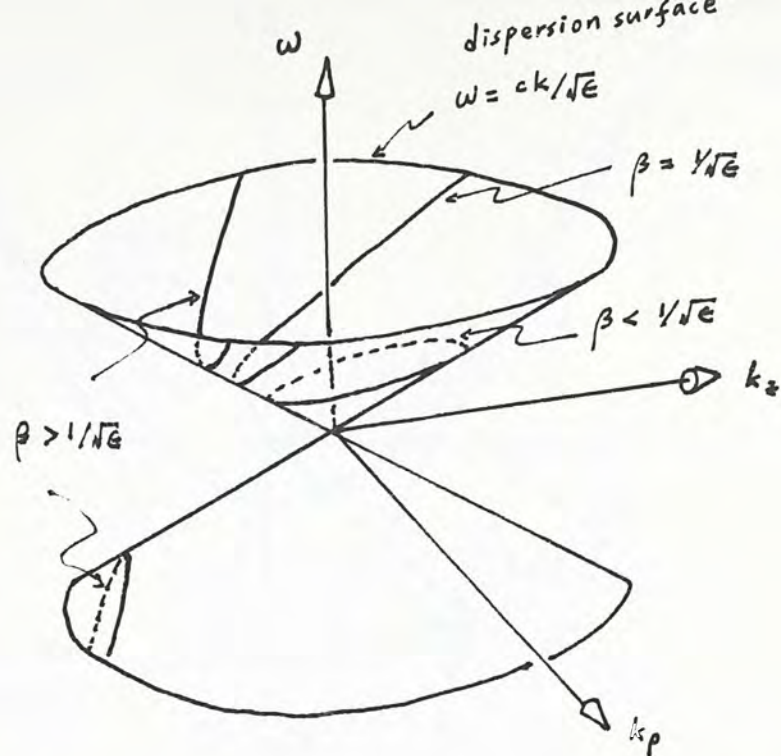
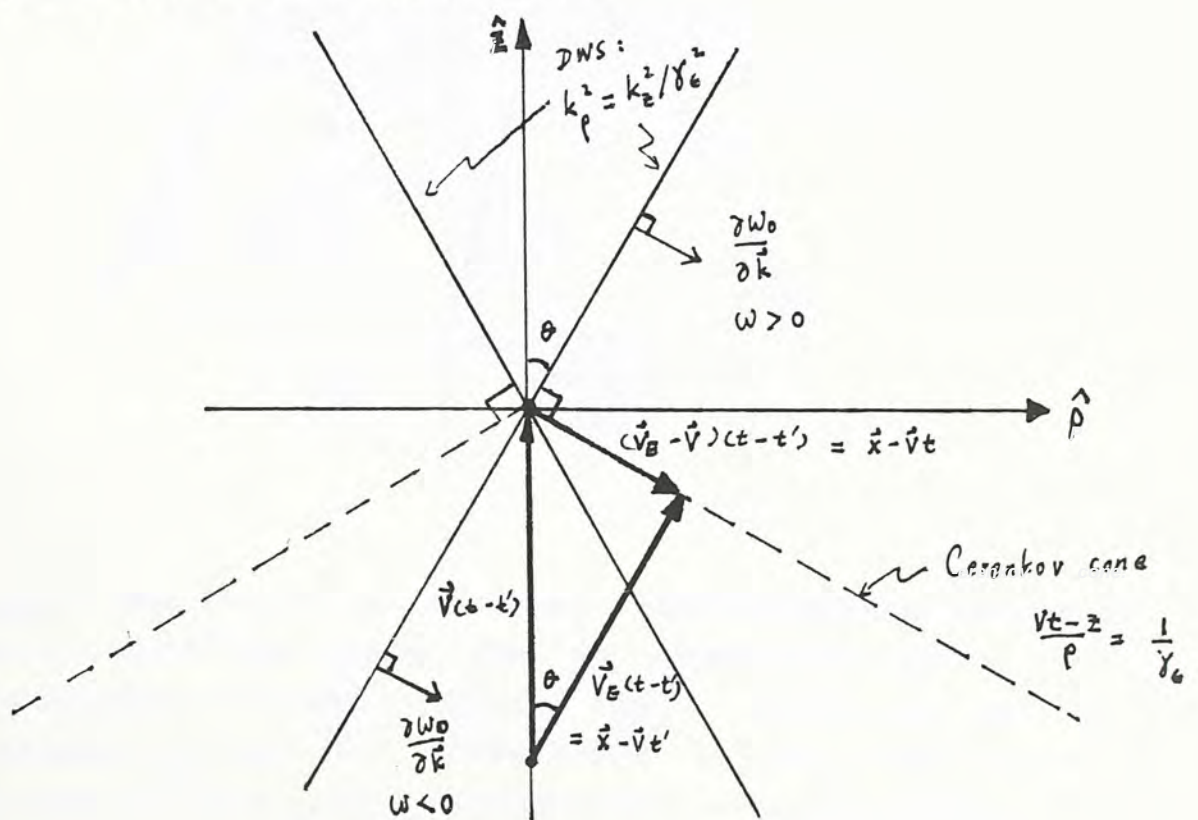


Fig. 3-5 DWS's of isotropic cold plasma, with  $n_0^2 \equiv 1 - (\omega_p / \omega_0)^2$ , are drawn for different  $\beta$ . Note that for  $\beta > n_0$ , the whole DWS is above the  $k_{\rho}$  axis and for  $\beta \rightarrow 1$ , the length of the DWS also tends to infinity but with both the highest and lowest points  $\rightarrow \omega^+$ .



**Fig. 6-1** The dispersion surface for an isotropic non-dispersive medium is a conical surface, the intersection of this surface with the Doppler plane may have three kinds of topologies : (a) for  $\beta < 1/\sqrt{\epsilon}$  it is an ellipsoid; (b) for  $\beta = 1/\sqrt{\epsilon}$  it is a paraboloid; (c) for  $\beta > 1/\sqrt{\epsilon}$ , it is a hyperboloid.



**Fig. 6-3** The DWS for the Cerenkov radiation in an isotropic non-dispersive medium is a conical surface. The normal of this surface points at one direction only (for one  $\phi$ ), so only the observation points lying on this direction can find radiation. This is just the surface of the Cerenkov cone.



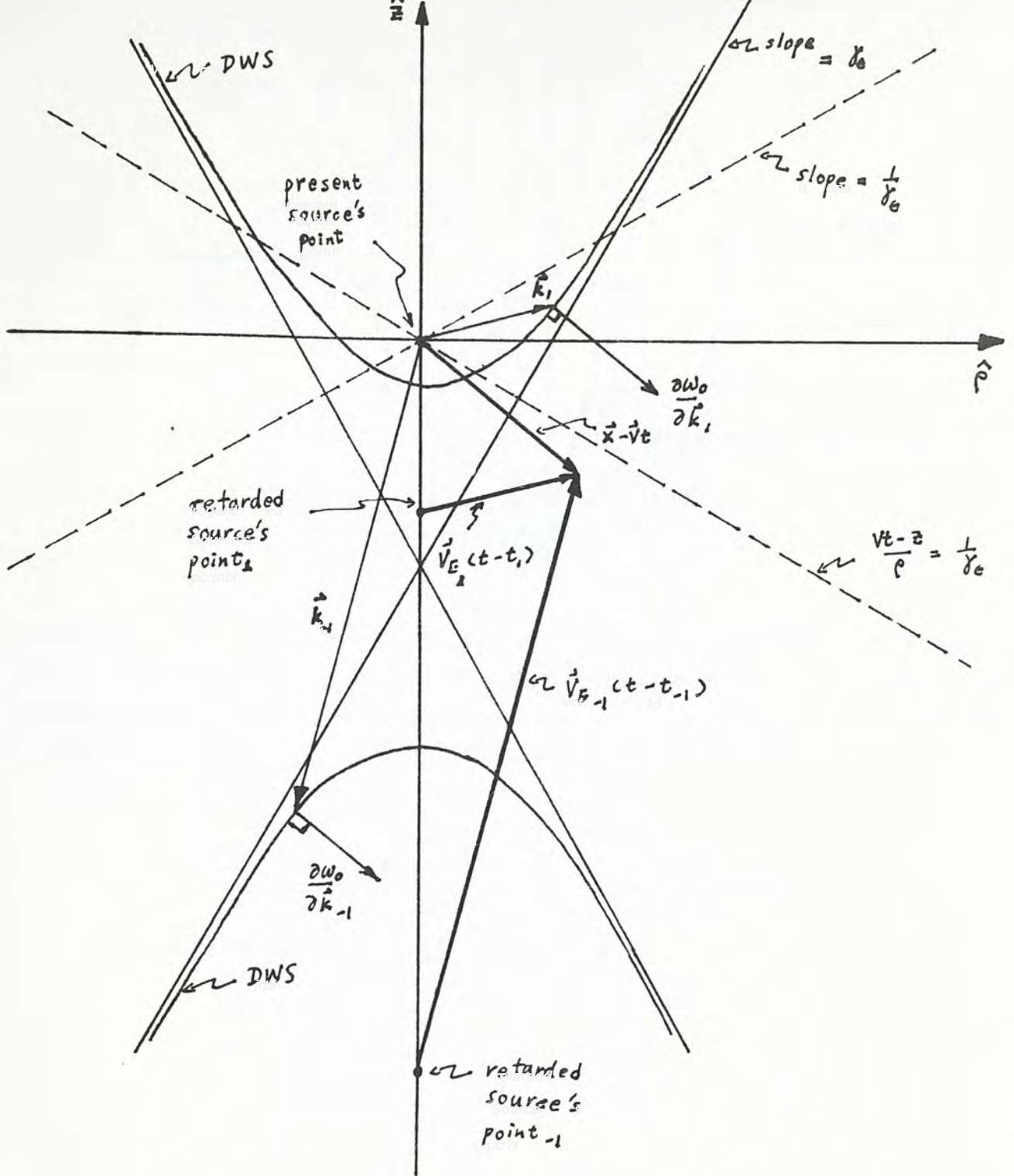


Fig. 6-2 The DWS of an isotropic non-dispersive medium for the case  $\beta > 1/\gamma_e$  is drawn. For an observation point inside the Cerenkov cone  $\tau \equiv (Vt - z)/\rho > 1/\gamma_e$ , there are two points of stationary  $k_1$  and  $k_{-1}$  corresponds to it. The two retarded positions  $t'_1$  and  $t'_{-1}$  of the source are also indicated.

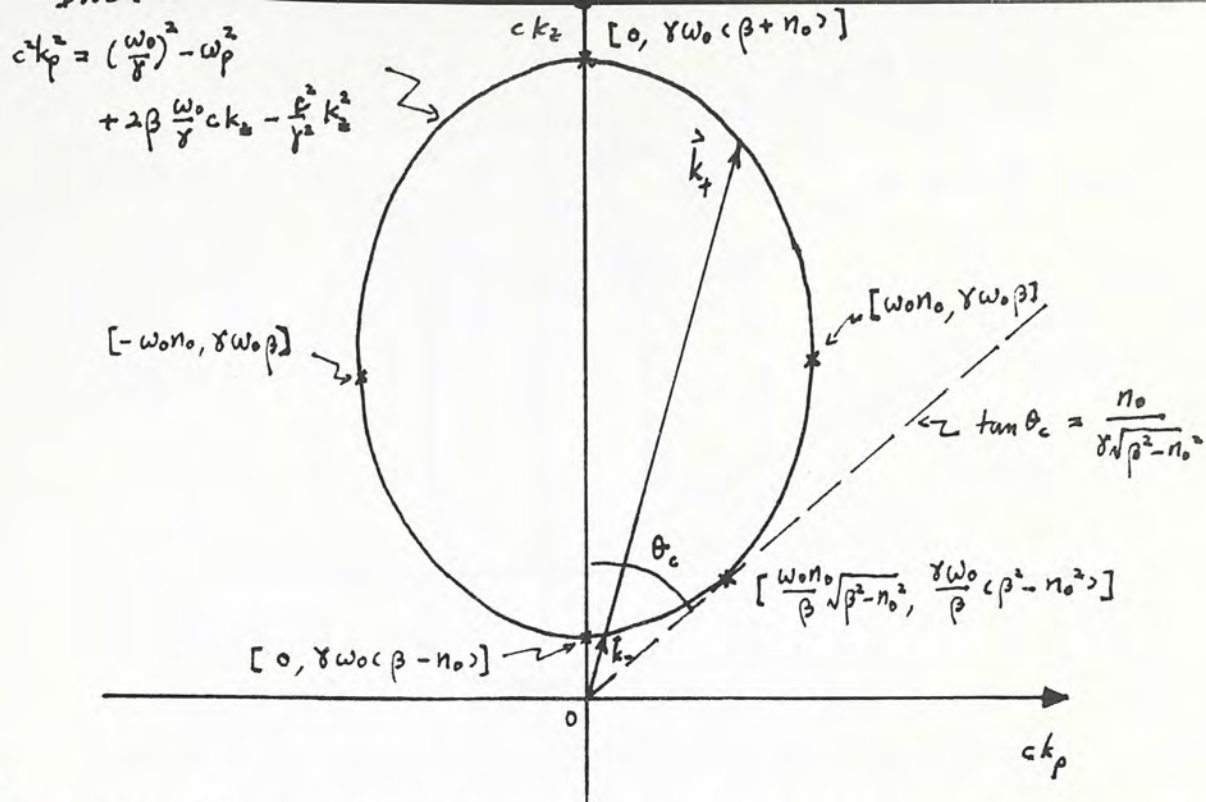


Fig. 7-1a The DWS for an isotropic cold plasma when  $\omega_0 < \gamma\omega_p$  only exist for  $\theta < \theta_c$ . For each  $\theta$  in this cone, there are two  $k$  values. Note also that the radiation only propagate in the forward direction of this cone.

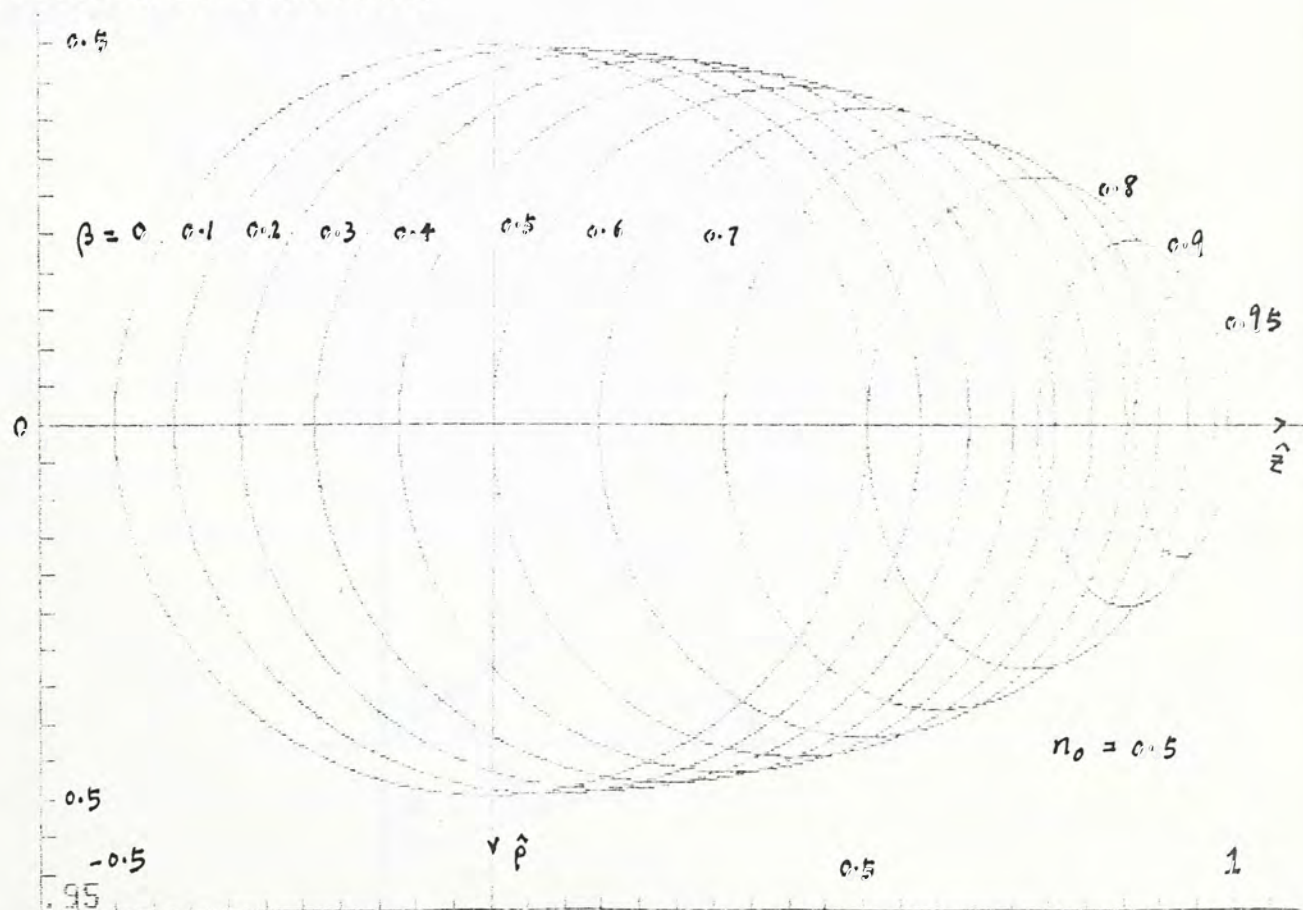
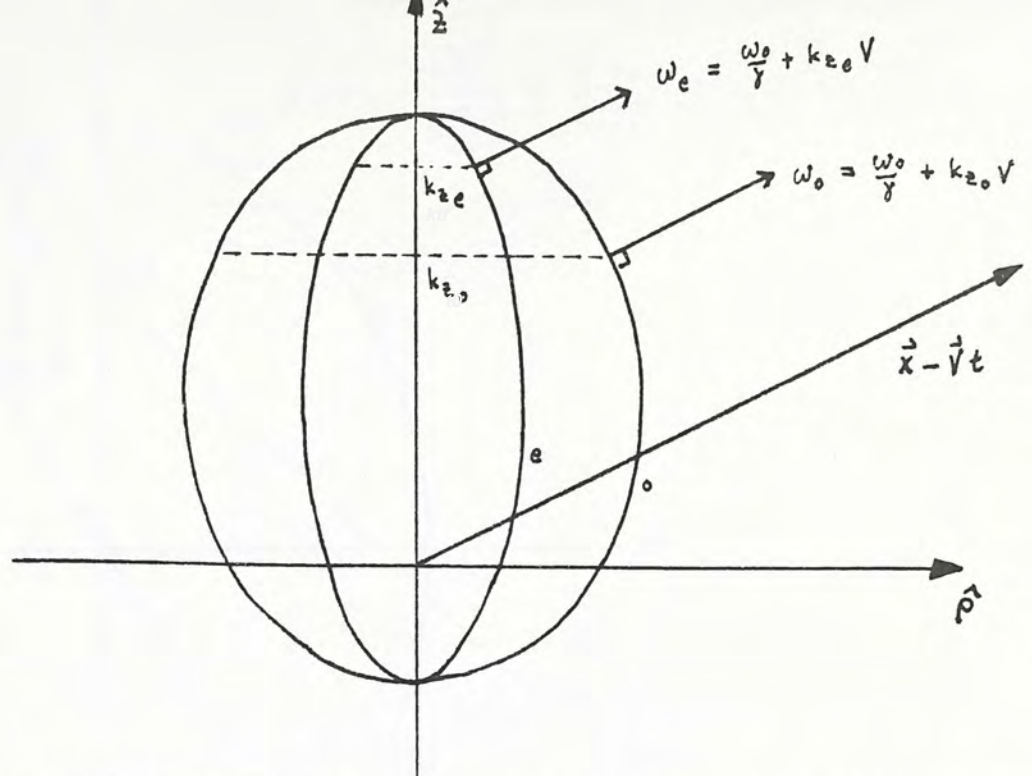


Fig. 7-1b Group velocity surface for an isotropic cold plasma, i.e. Plotting  $V_E$  as a function of  $\theta$  :

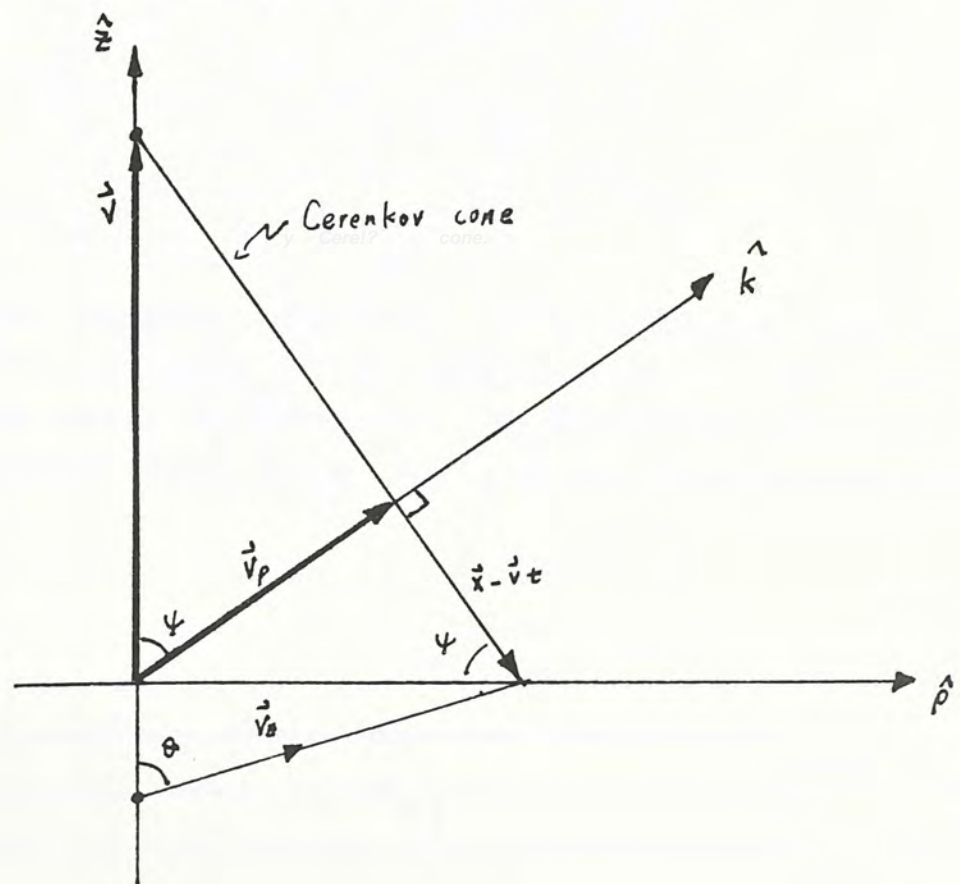
$$\beta_E \equiv V_E / c$$

$$= \frac{\beta \cos \theta (\gamma \omega_p / \omega_0)^2 \pm \left[ 1 - (\gamma \omega_p / \omega_0)^2 (1 - \beta^2 \cos^2 \theta) \right]}{1 + (\gamma \omega_p / \omega_0)^2 \beta^2 \cos^2 \theta}$$

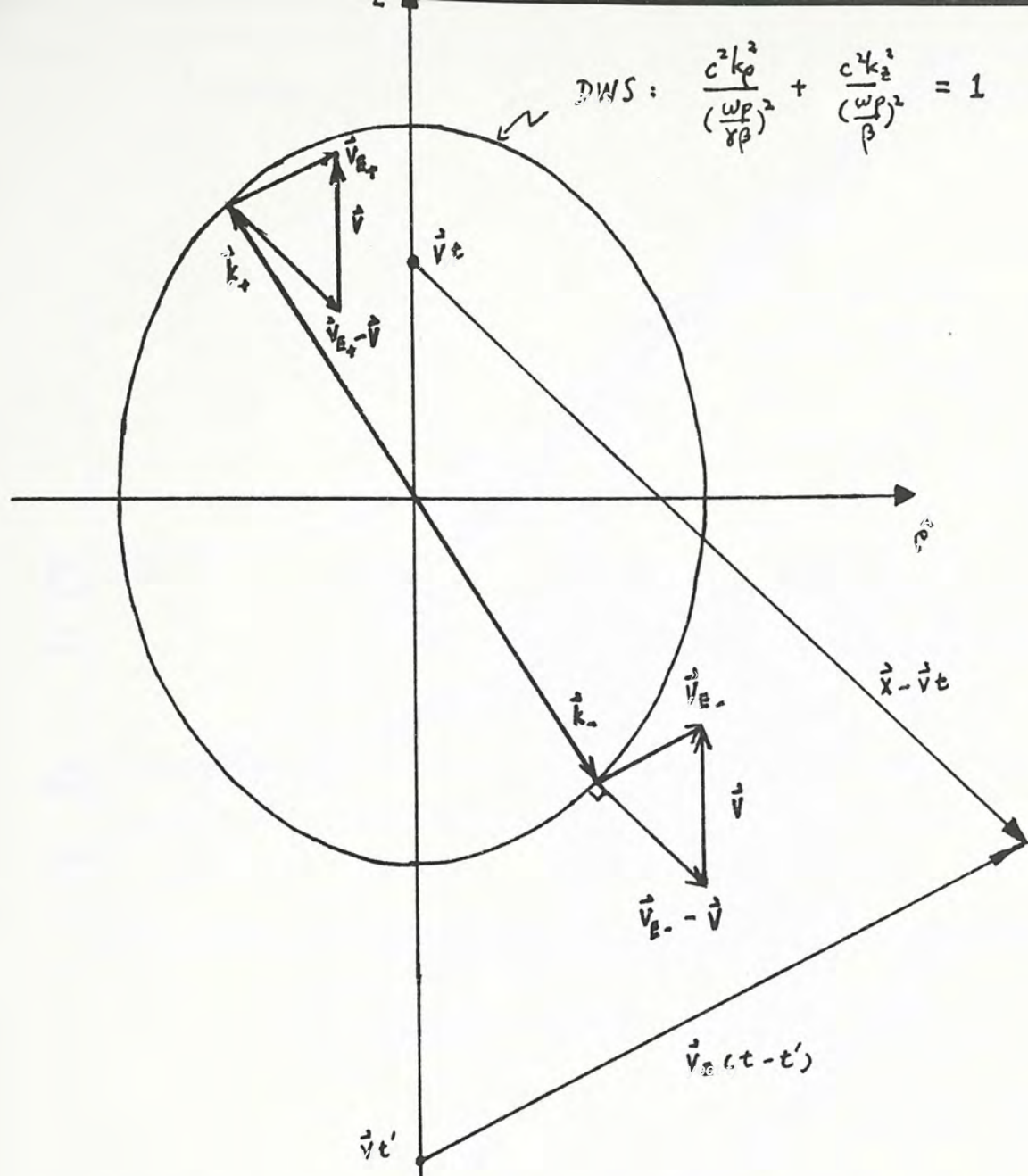




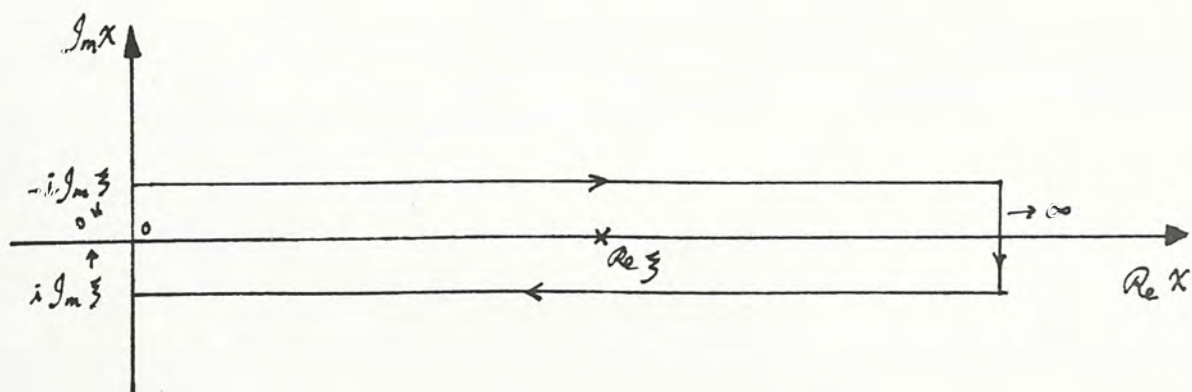
**Fig.8-1** DWS for a non-dispersive uniaxial medium are drawn for the velocity of the source along the symmetric axial  $\hat{z}$ . The frequency of the received waves from the (o) and (e) mode for a observation point is not the same in general, except for  $x - Vt$  parallel or perpendicular to  $\hat{z}$ .



**Fig.8-2** Cerenkov cone for the (e) mode. Note that  $k \perp x - Vt$  on the surface of the cone.



**Fig. 9-1** DWS for Cerenkov radiation in an uniaxial cold plasma with  $\mathbf{V}$  along  $\mathbf{B}$ . Note that it is symmetric in  $\hat{z}$ ,  $-\hat{z}$ . From the direction of the vector  $\mathbf{V}_E$  and  $\mathbf{V}_E - \mathbf{V}$ , we found that there are two points of stationary phase  $k_+$ ,  $k_-$  for a given observation point  $\mathbf{x} - \mathbf{V}t$ .



**Fig. 9-2** The contour of integration.



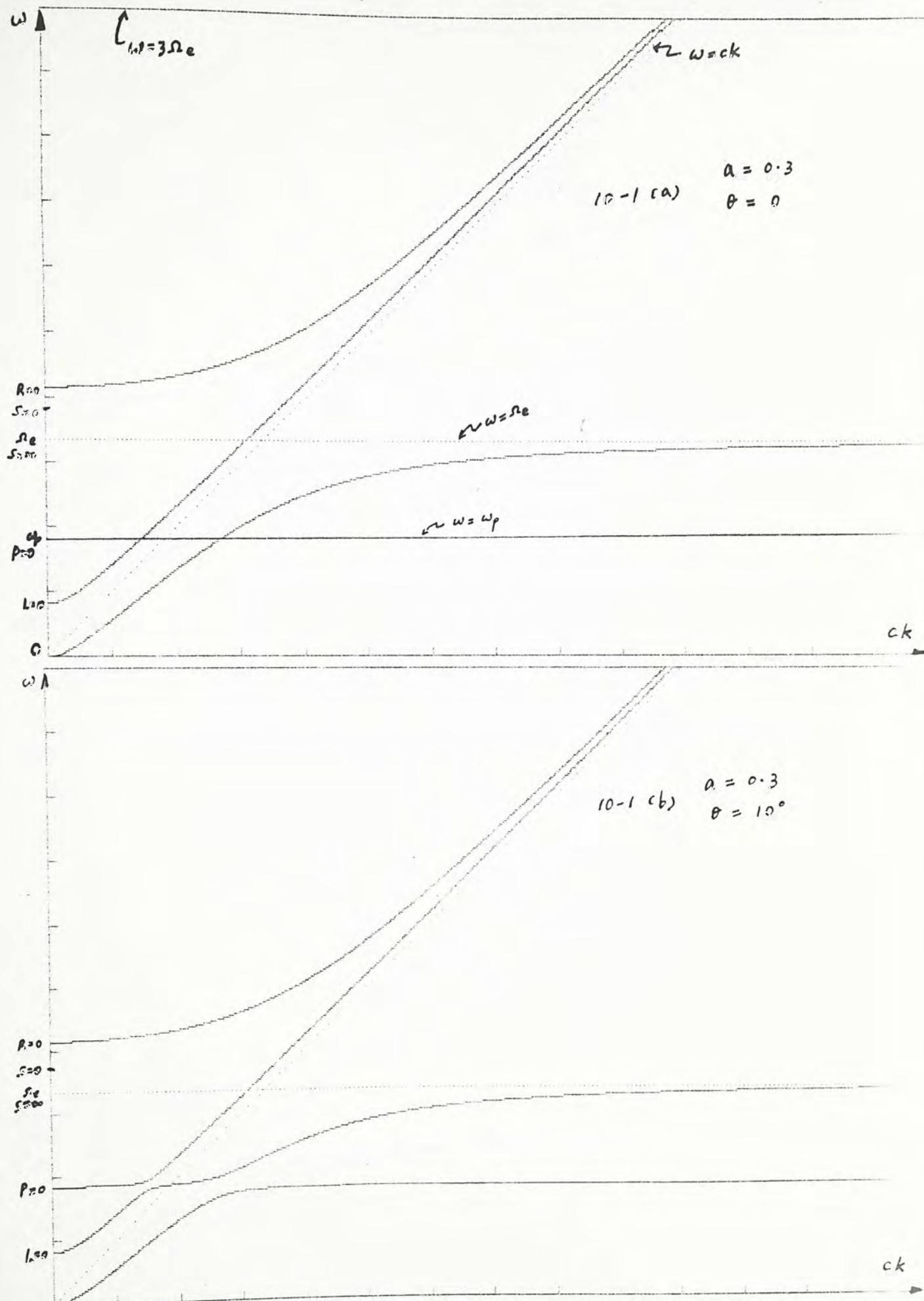
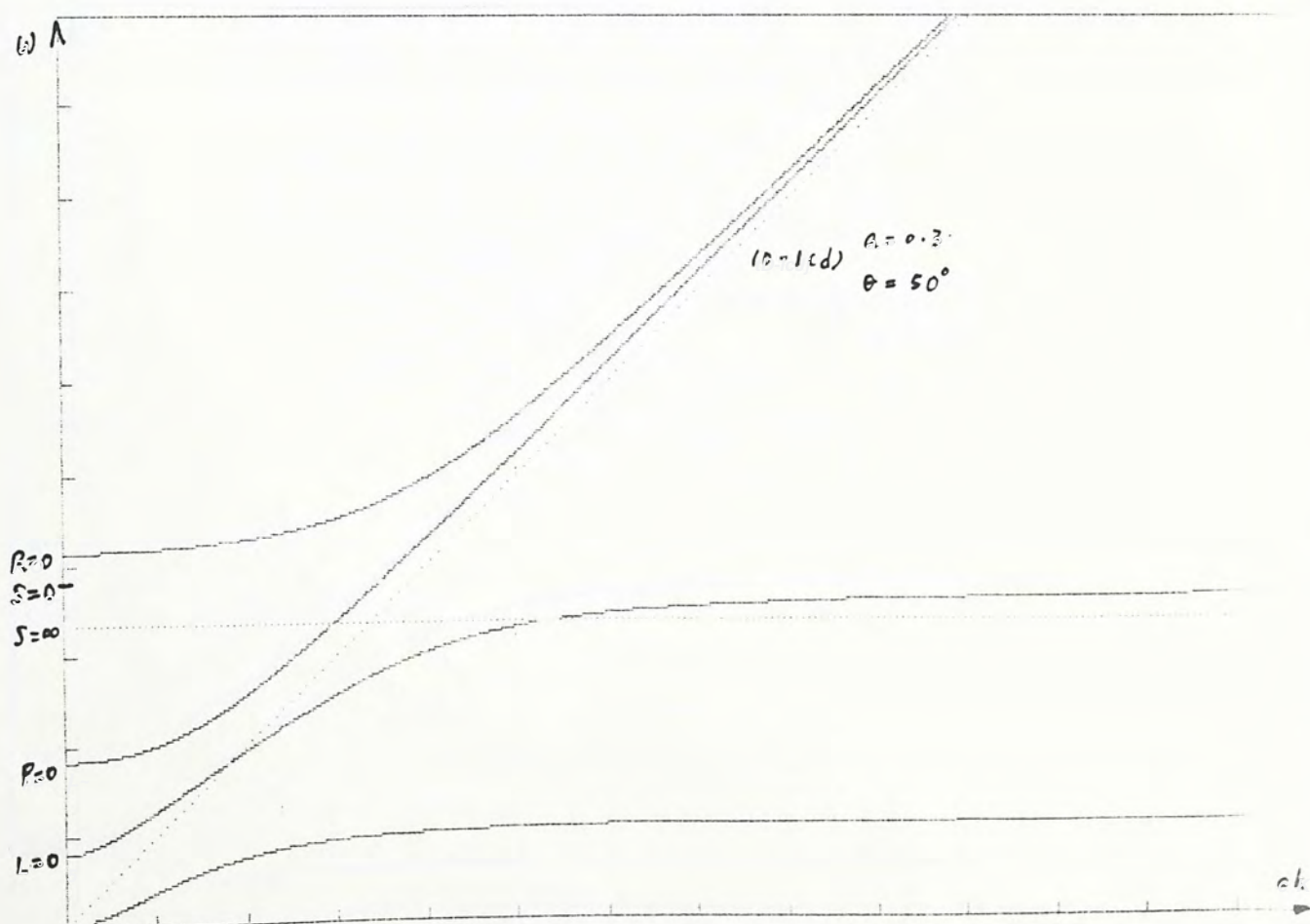
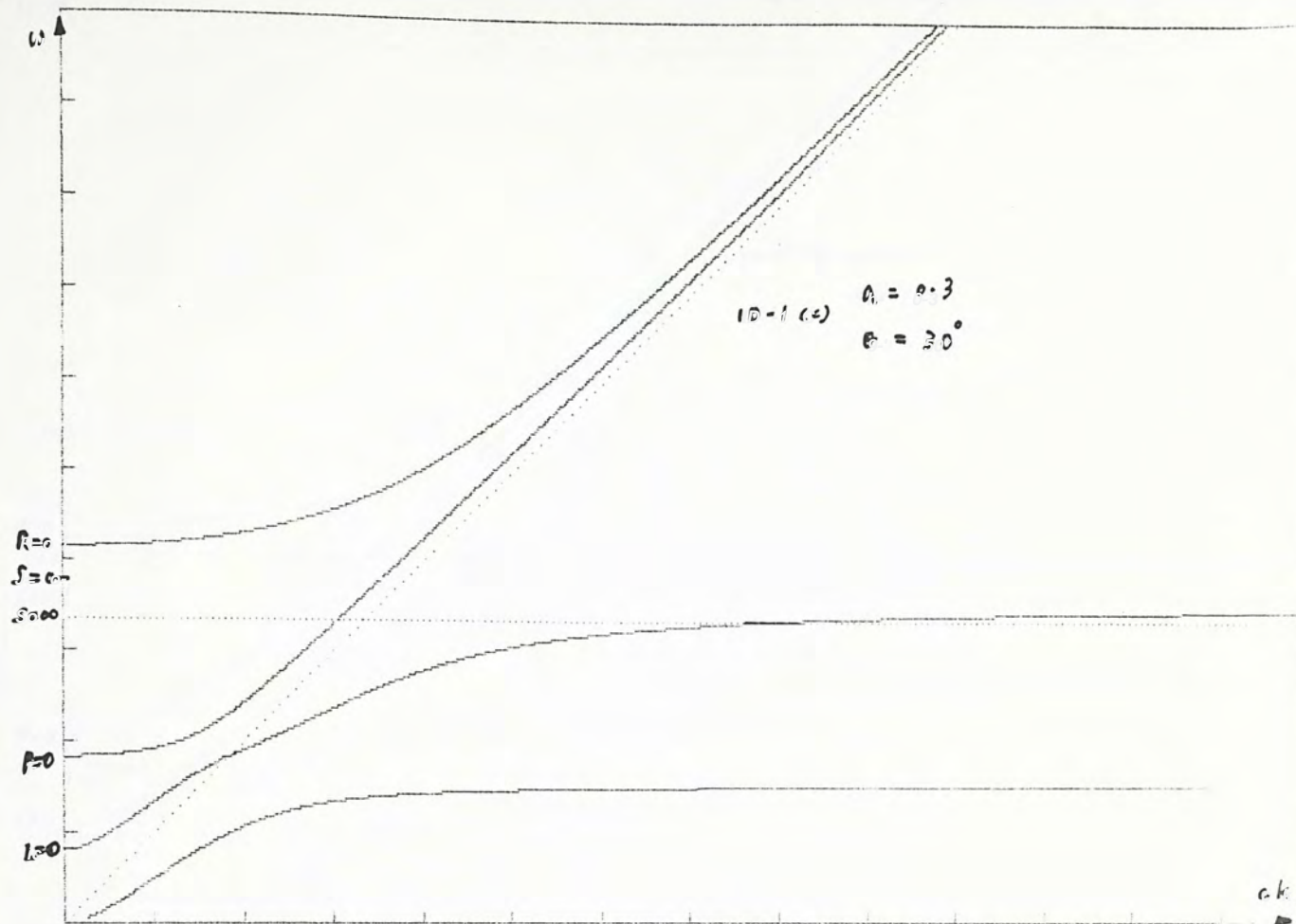
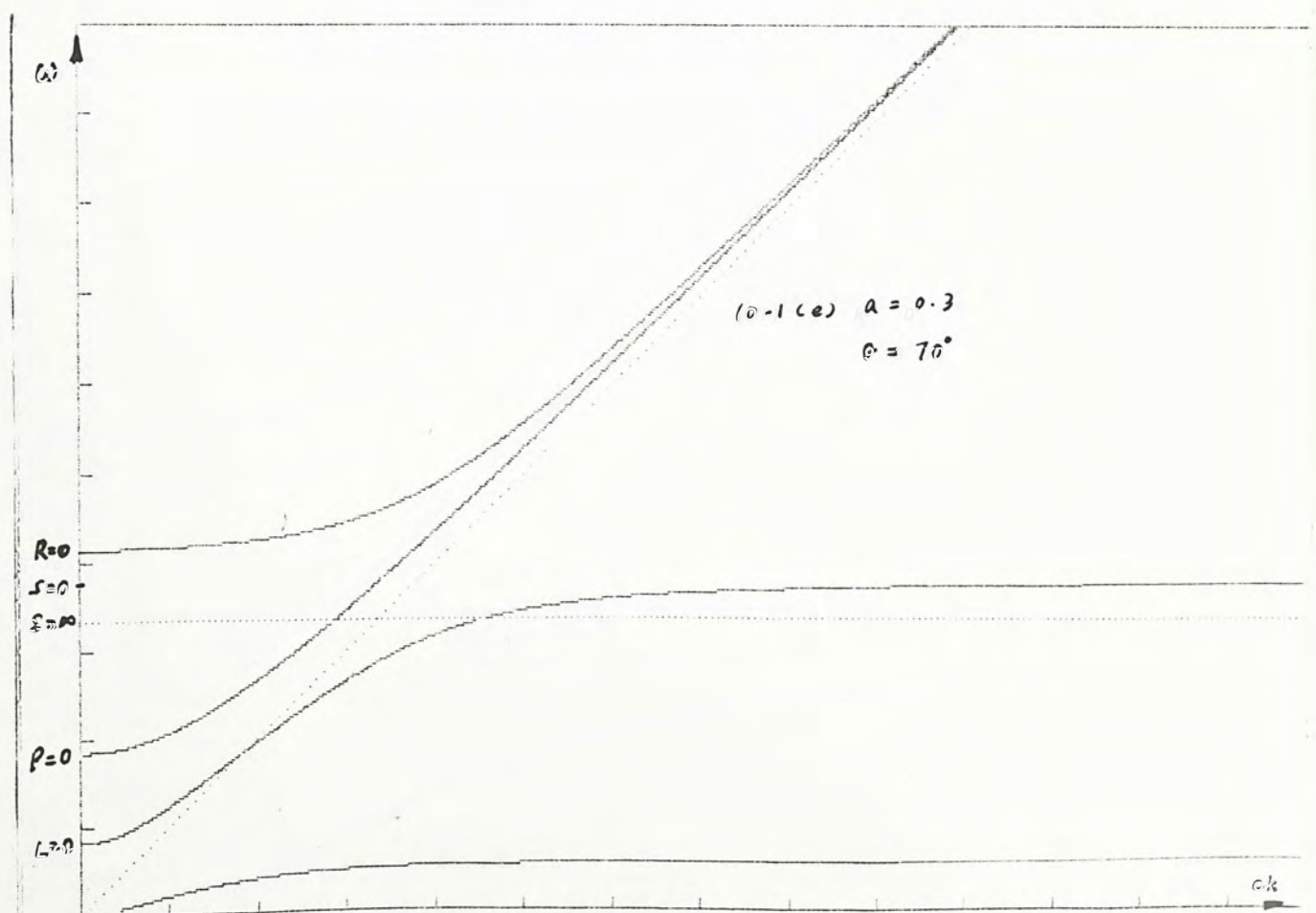
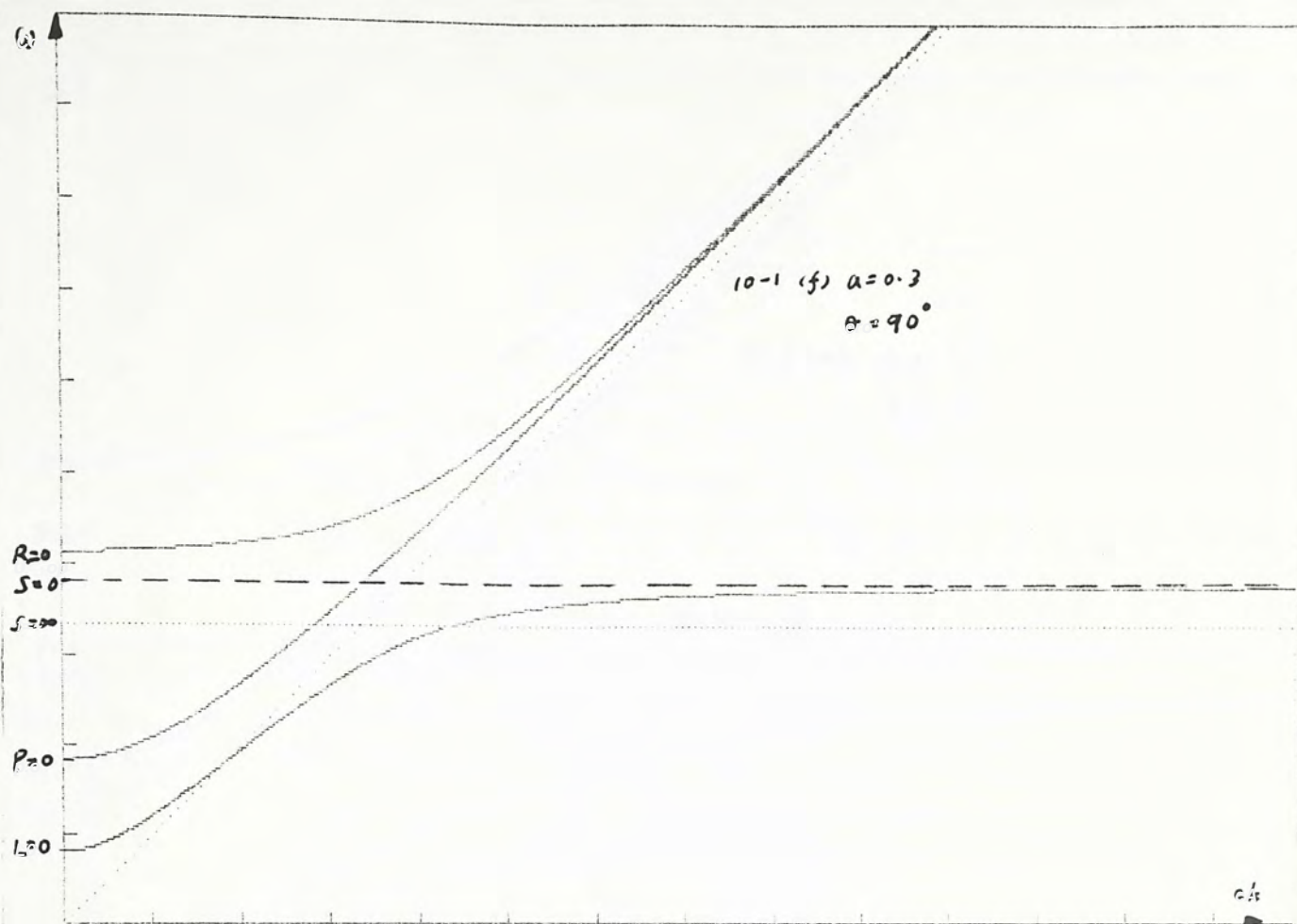
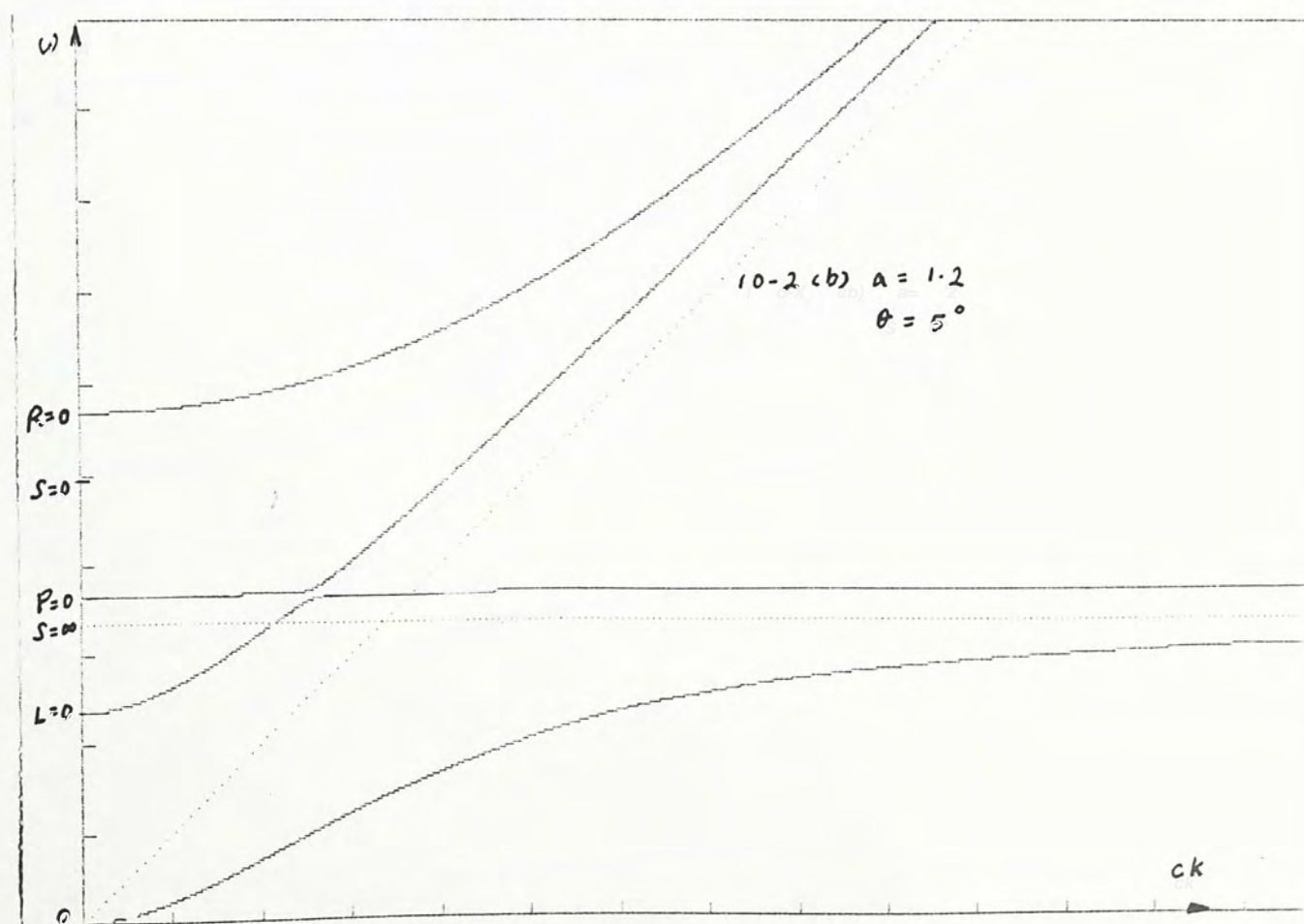
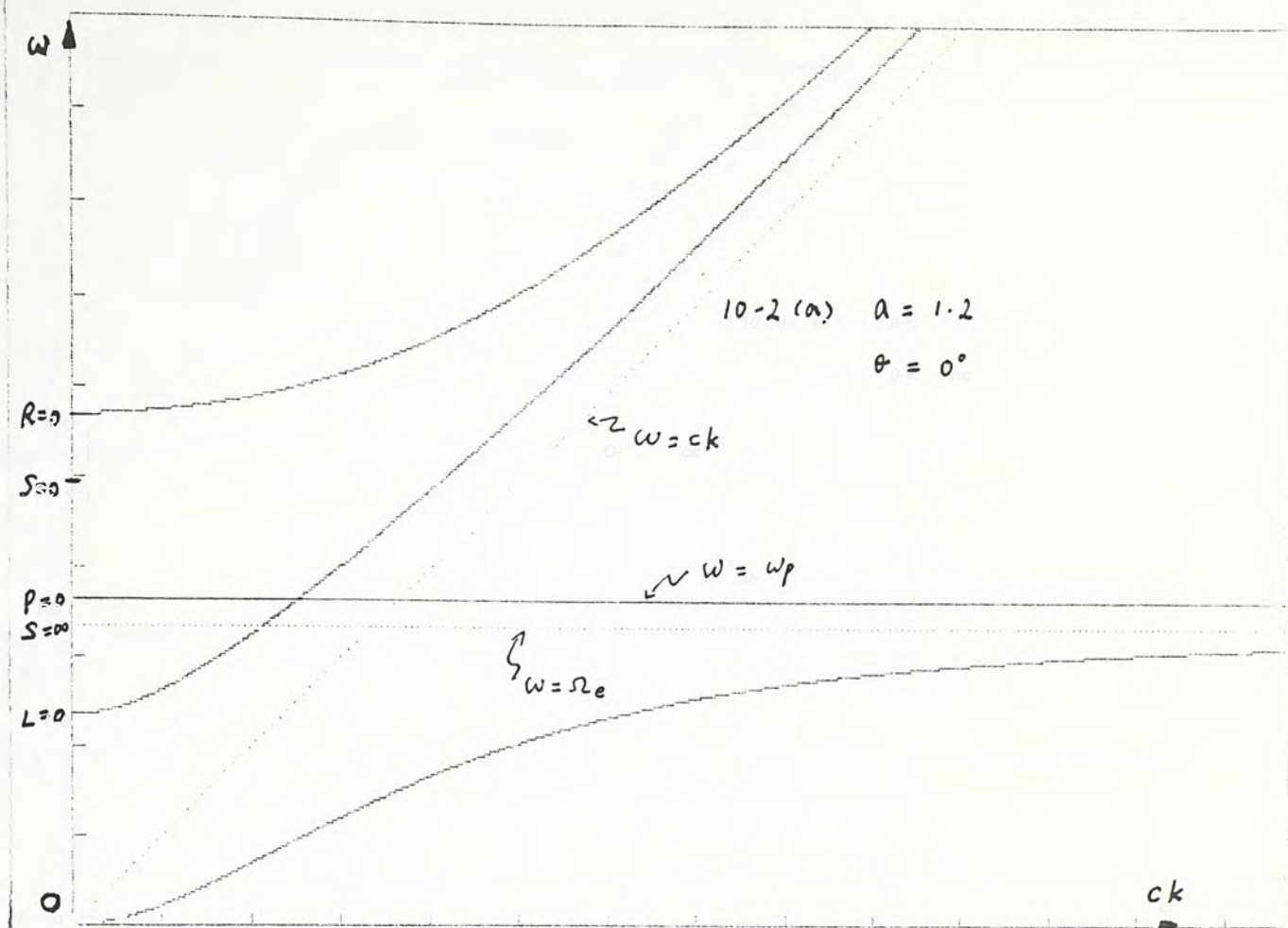


Fig.10-1 and Fig.10-2 The dispersion surfaces of a cold magnetoplasma are plotted as  $\omega$  vs  $k$  for different  $\theta$  (angle between  $k$  and the external B field) for two cases  $a = (\omega_{pe}/\Omega_e)^2 = 0.3$  and  $1.2$ . On the  $\omega$  axis, the values of the functions of  $\omega$ , S, P, D, R, L are indicated.

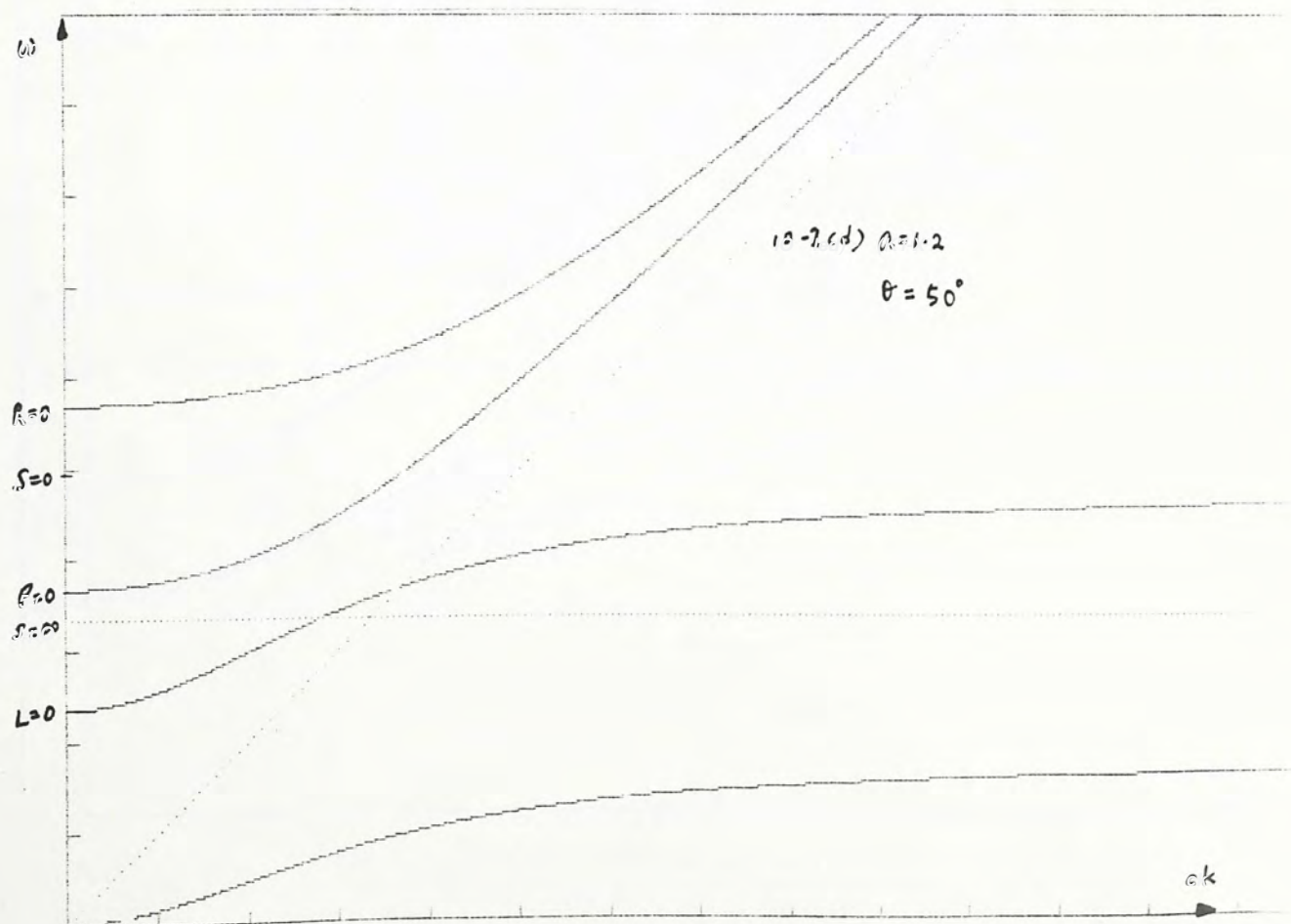
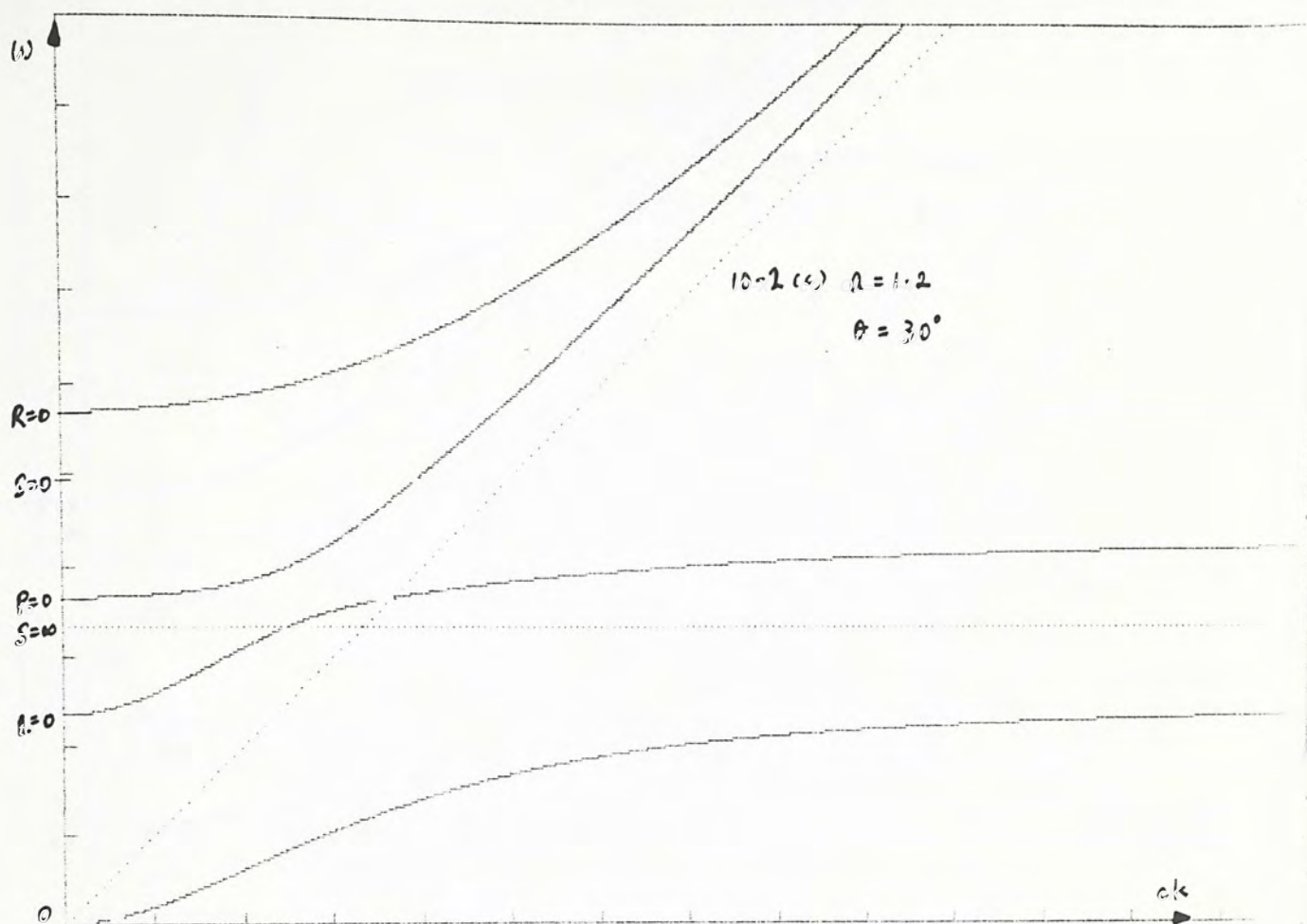


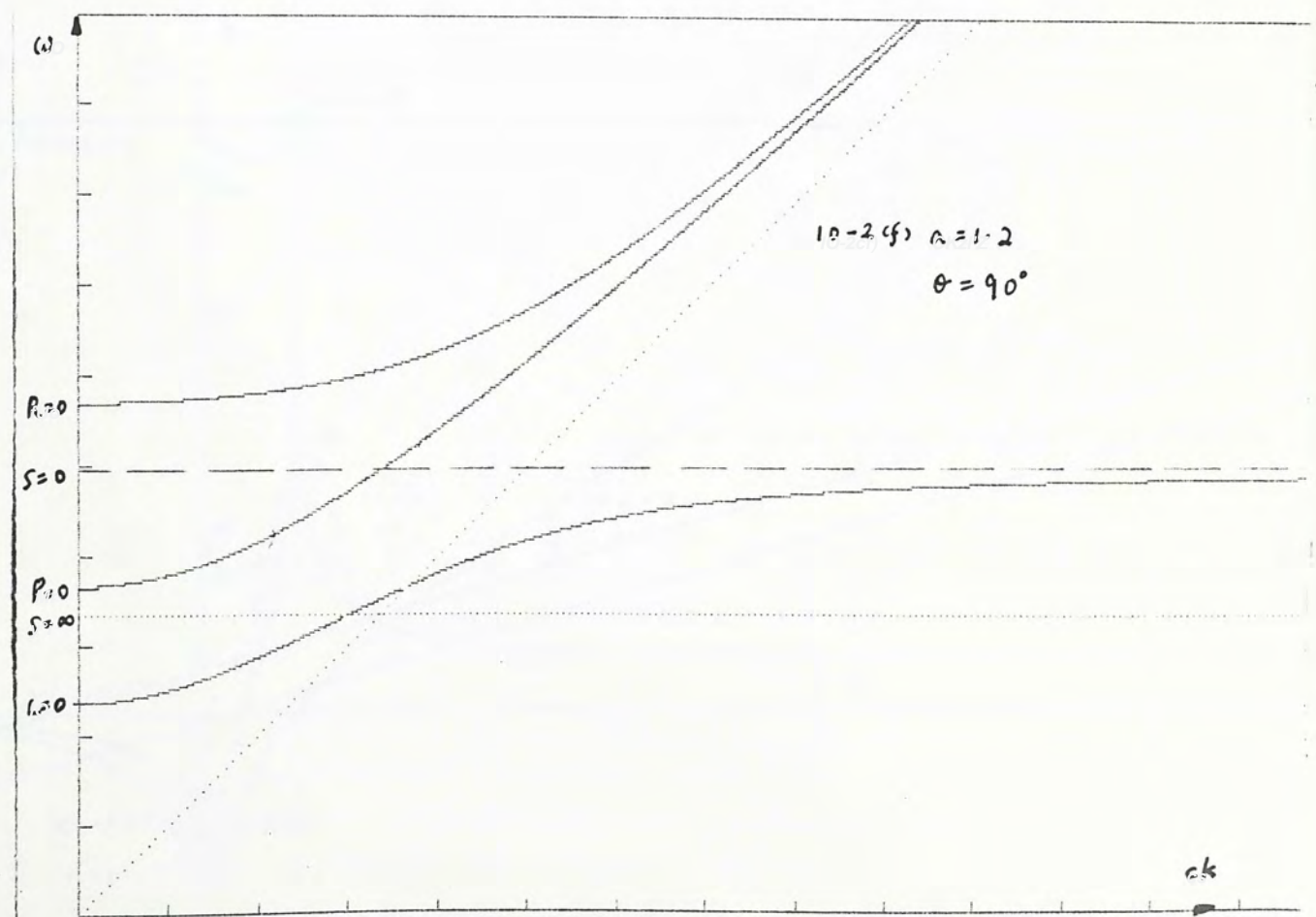
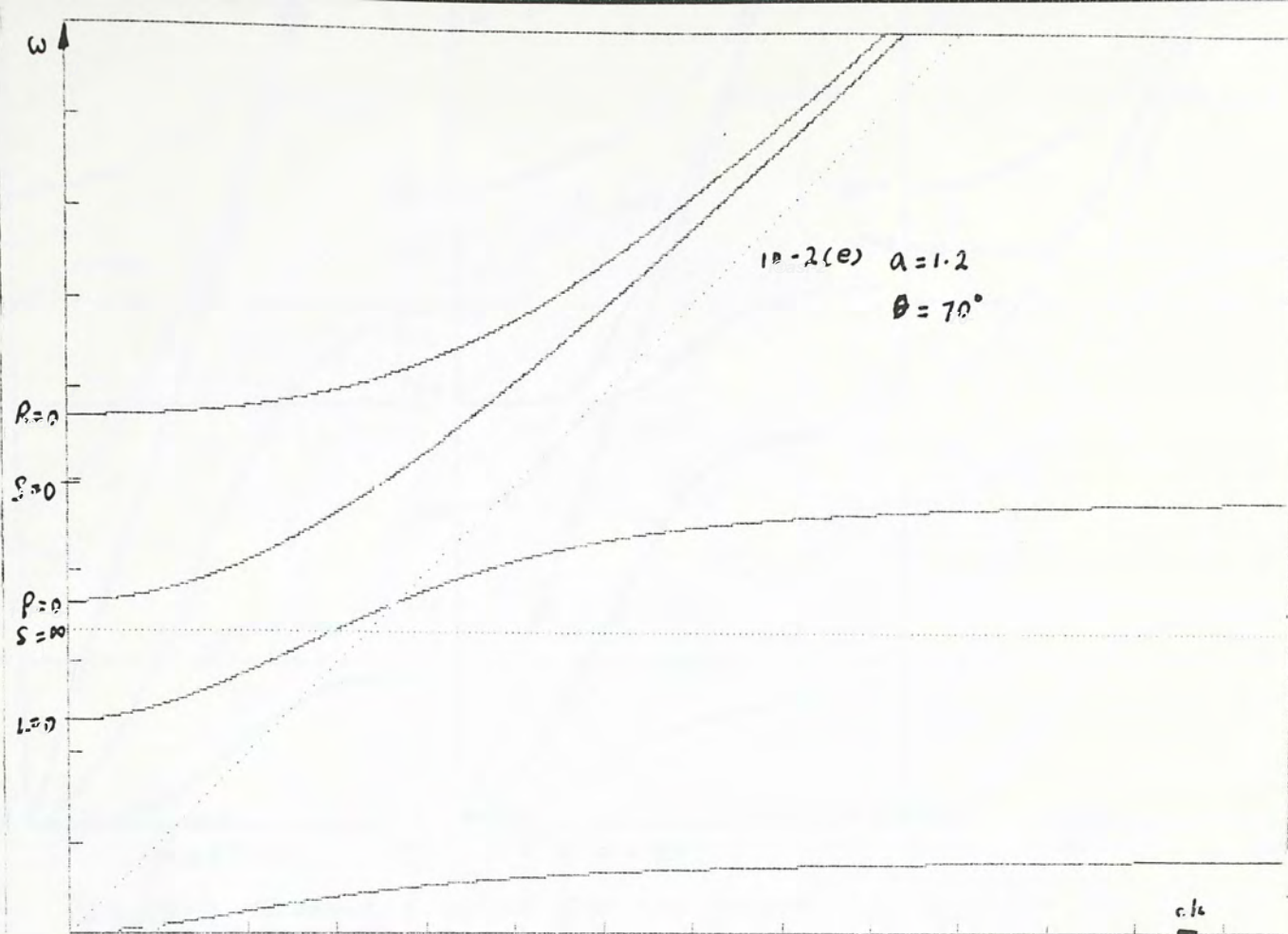














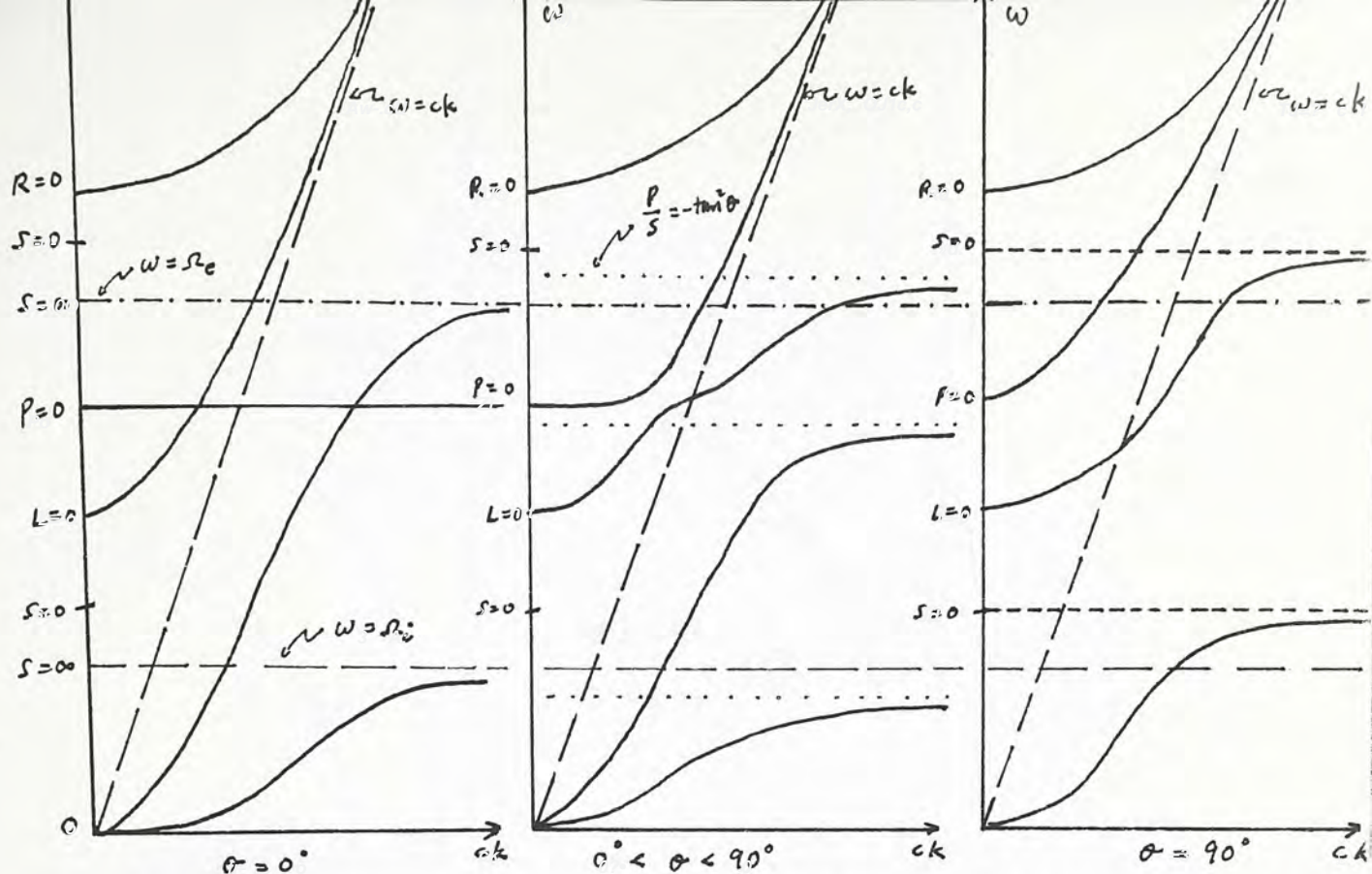


Fig.10-3 Schematic plots for the dispersion surface for  $\theta = 0$ ,  $\pi/2$  and a general  $\theta$ .

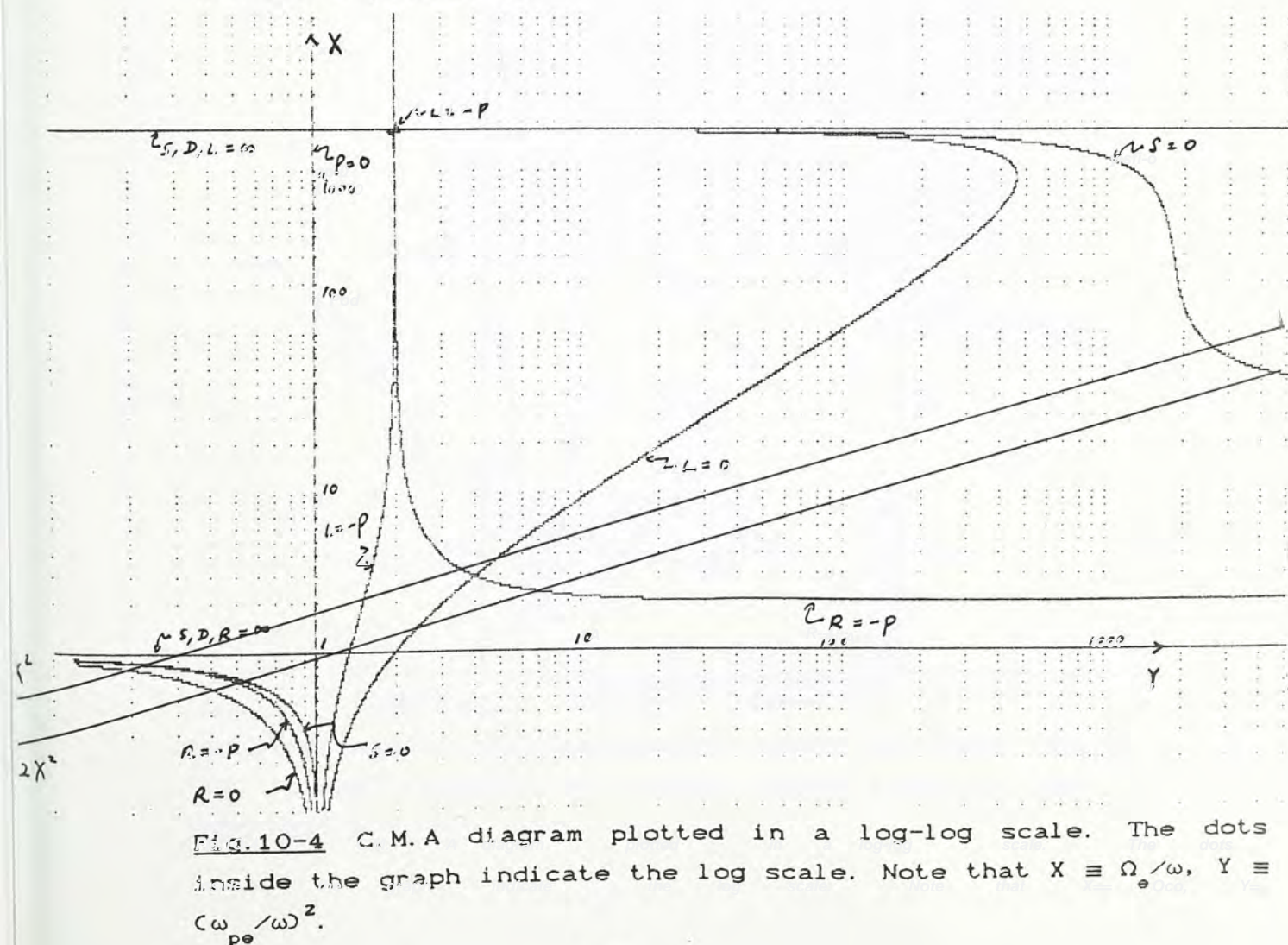


FIG.10-4 C.M.A diagram plotted in a log-log scale. The dots inside the graph indicate the log scale. Note that  $X \equiv \Omega_0/\omega$ ,  $Y \equiv (C\omega_0/\omega)^2$ .

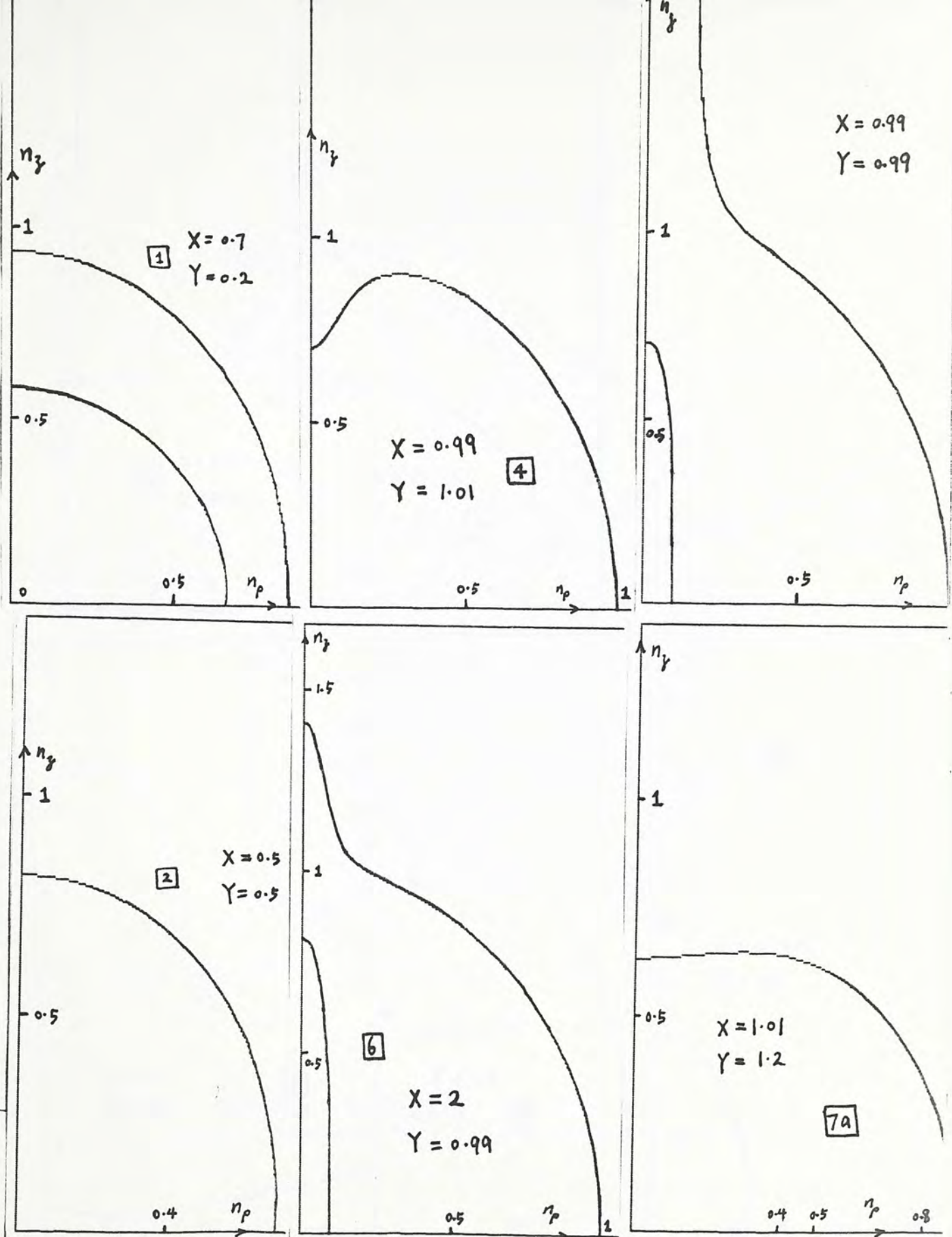
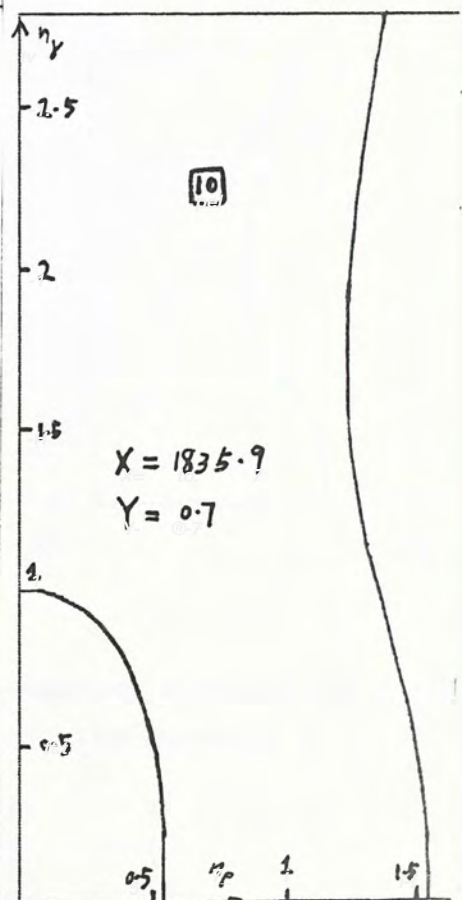
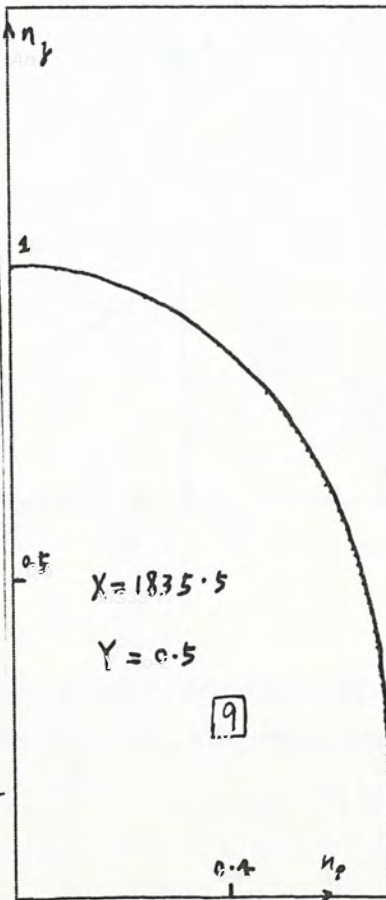
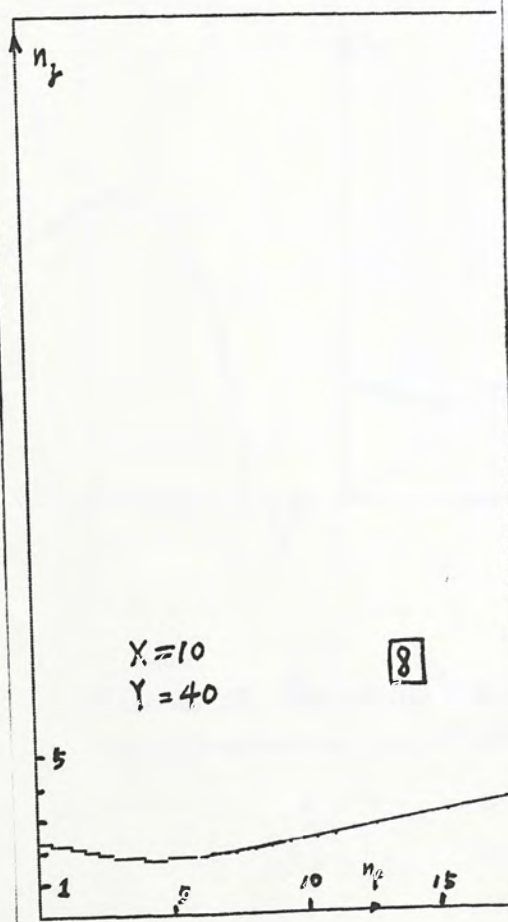
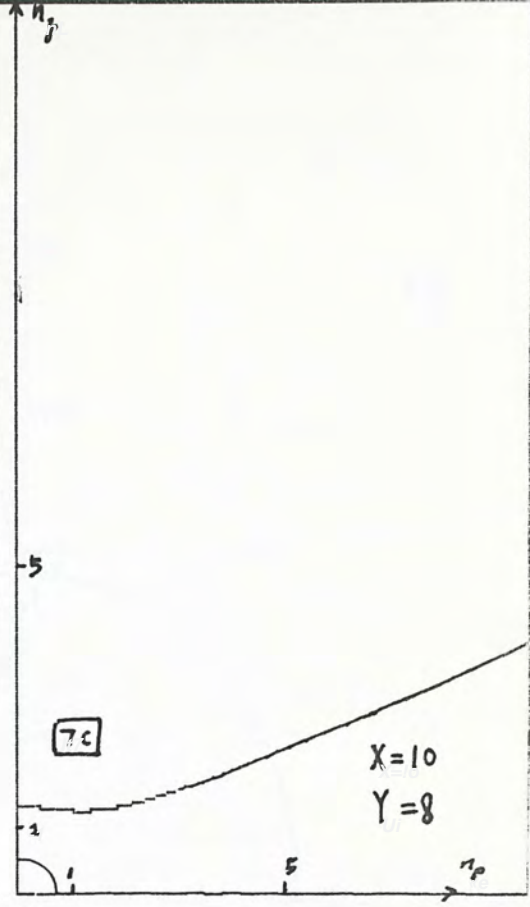
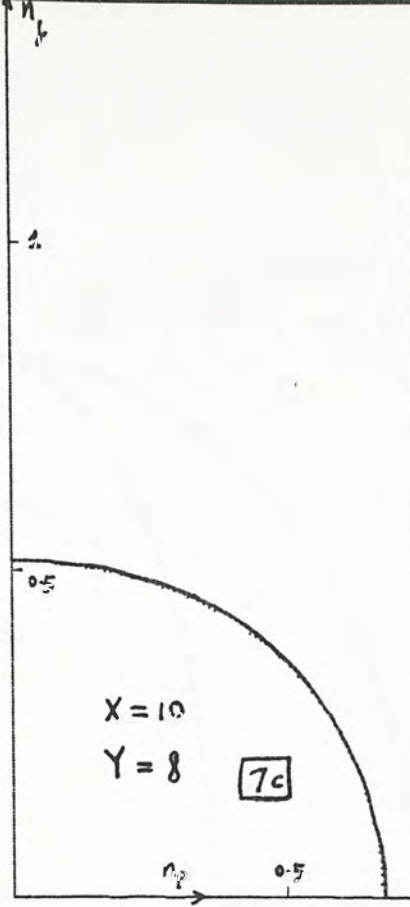
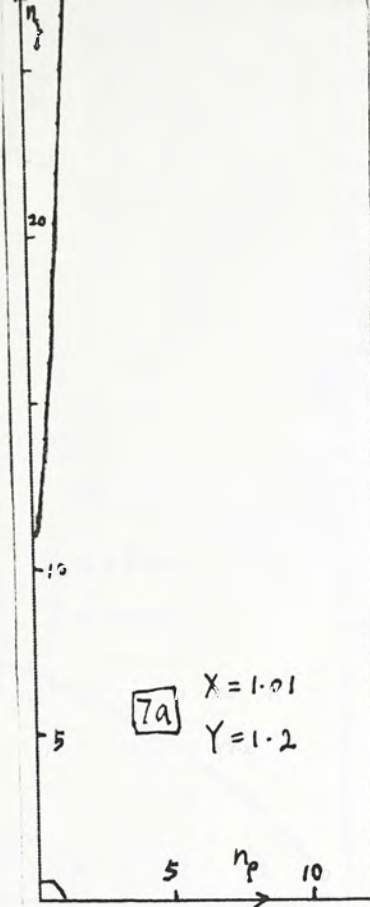


Fig.10-5 Wave-vector surface (DWS for  $\beta = 0$ ) of cold magnetoplasma for  $X, Y$  parameters inside different regions of the CMA diagram (indicate by a number inside a square, see Fig.10-7). Note that  $n \equiv ck/\omega$ .





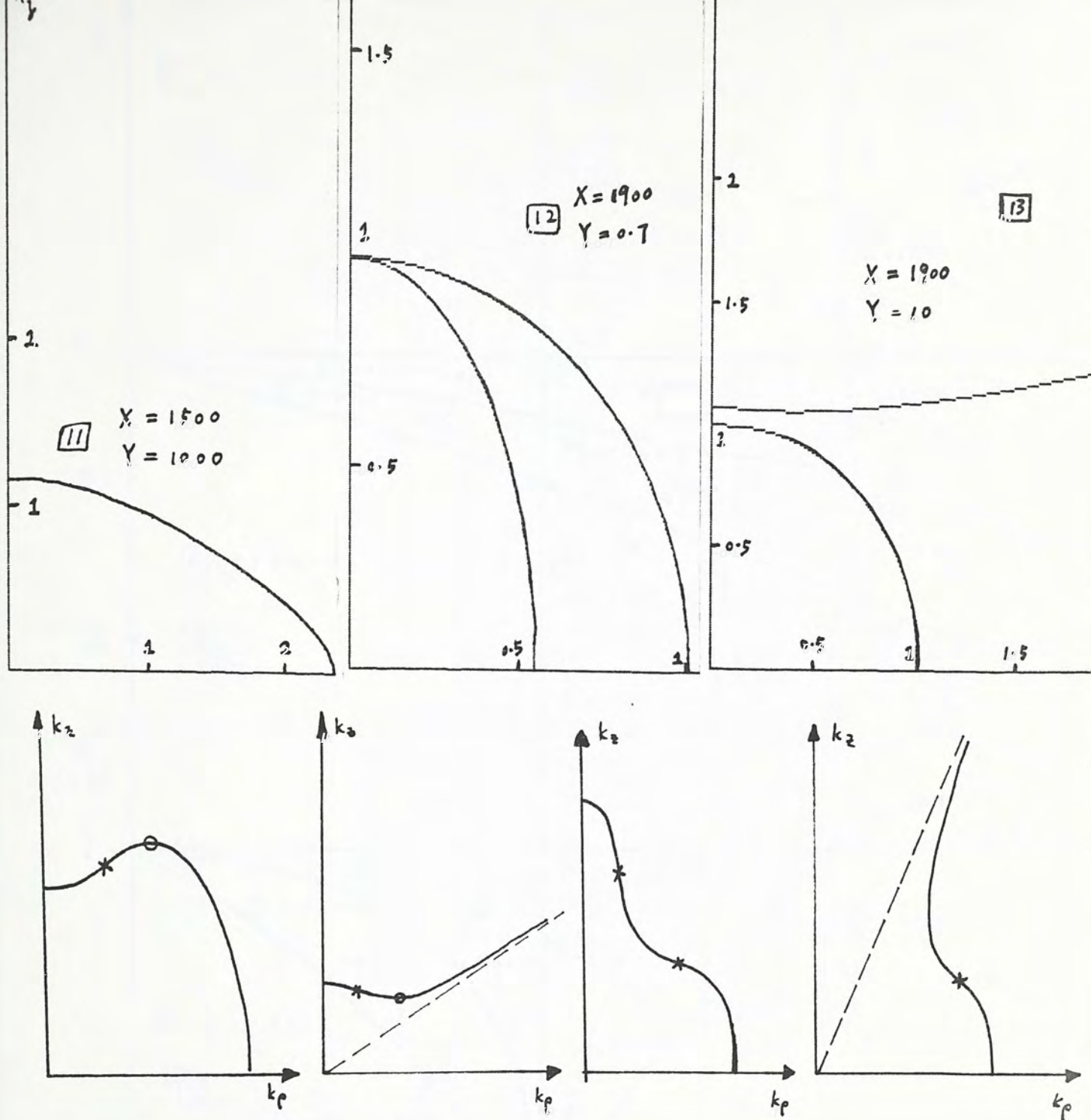


Fig. 10-6 Several cases found in the wave-vector surface of cold magnetoplasma such that  $K = 0$ . The meaning of the symbols are :

$$* : \lambda_{kz} = 0$$

$$o : \lambda_{\phi} = 0$$



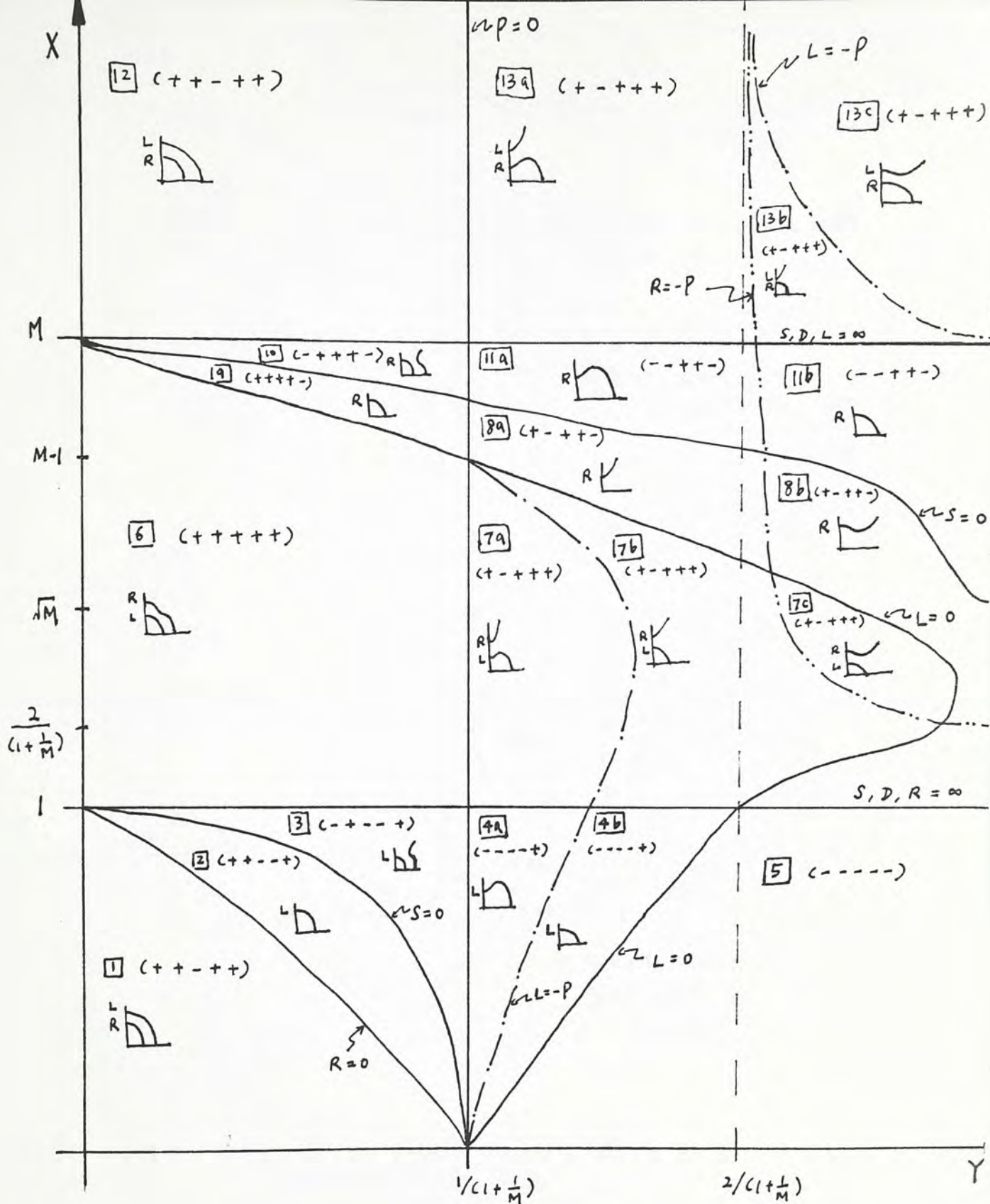



Fig.10-7 Schematic plots of the CMA diagram shows the separation of the parameter space. Note that  $R = -P$  and  $L = -P$  curve separate the regions whether there exist  $\lambda_\phi = 0$  case or not. The  $(+, -, +, +, -)$  indicates the sign of parameter  $(S, P, D, R, L)$ . The picture  is the schematic plot of wave-vector with  $R, L$  meaning  $n^2 = R$  or  $n^2 = L$  at  $\theta = 0$ .

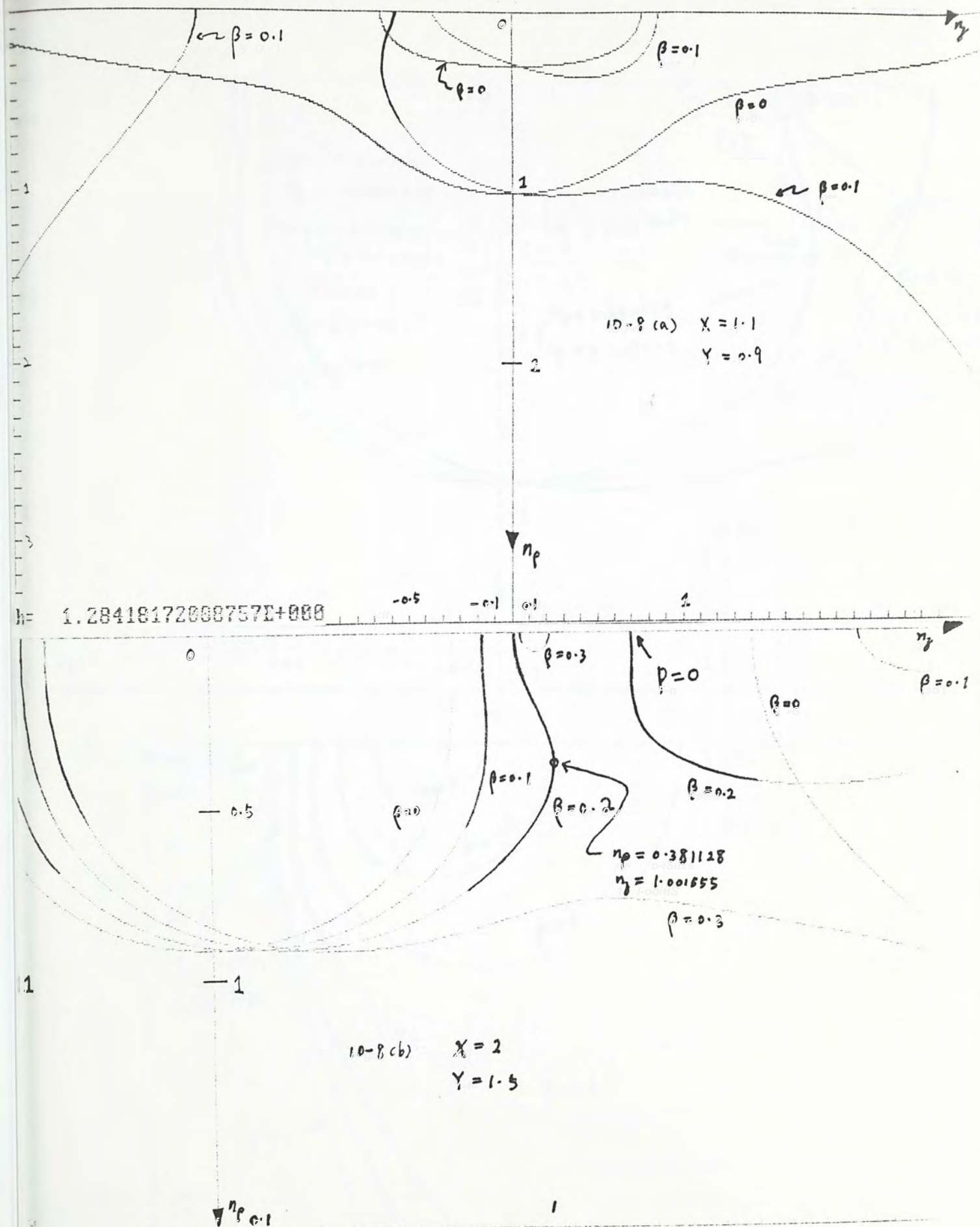
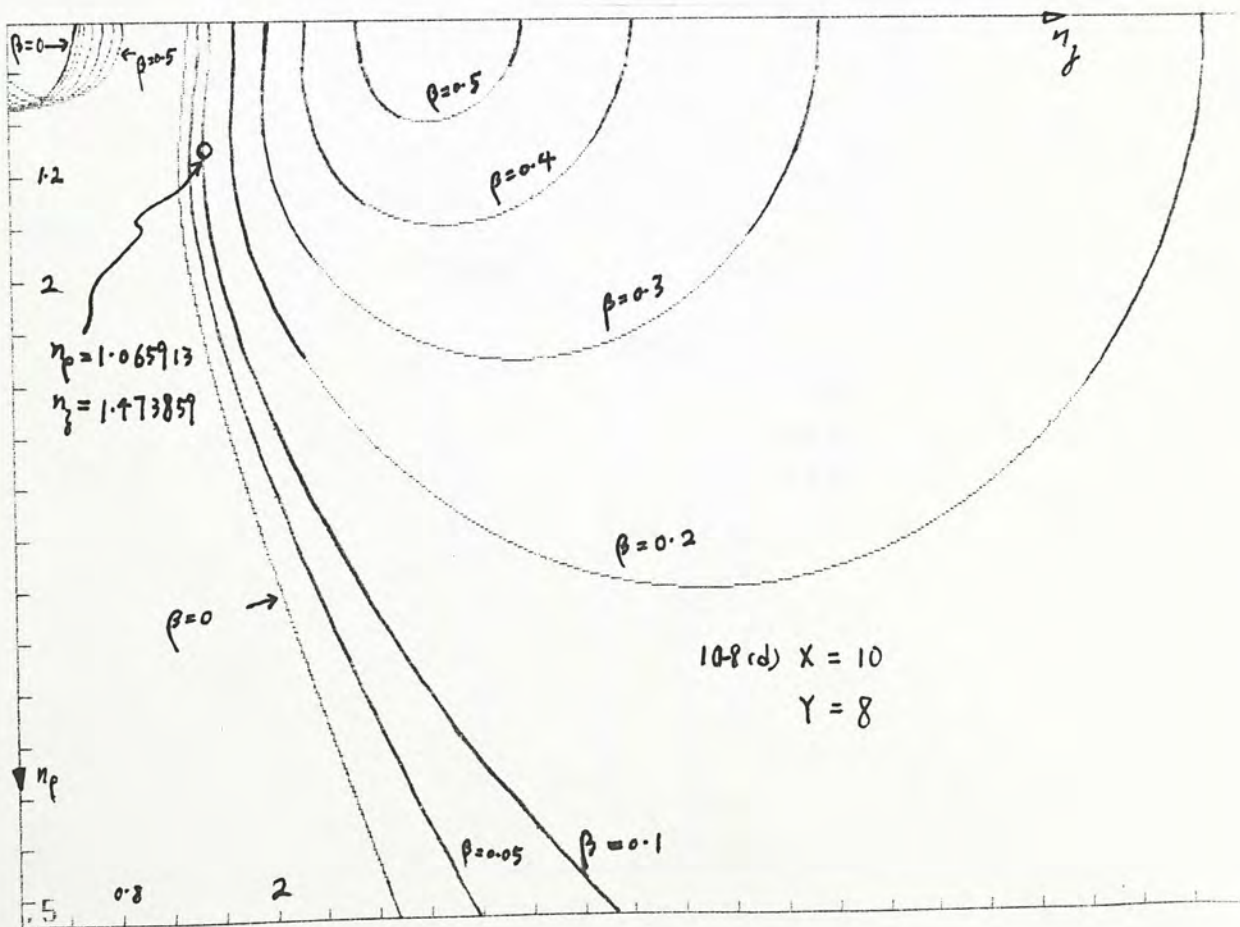
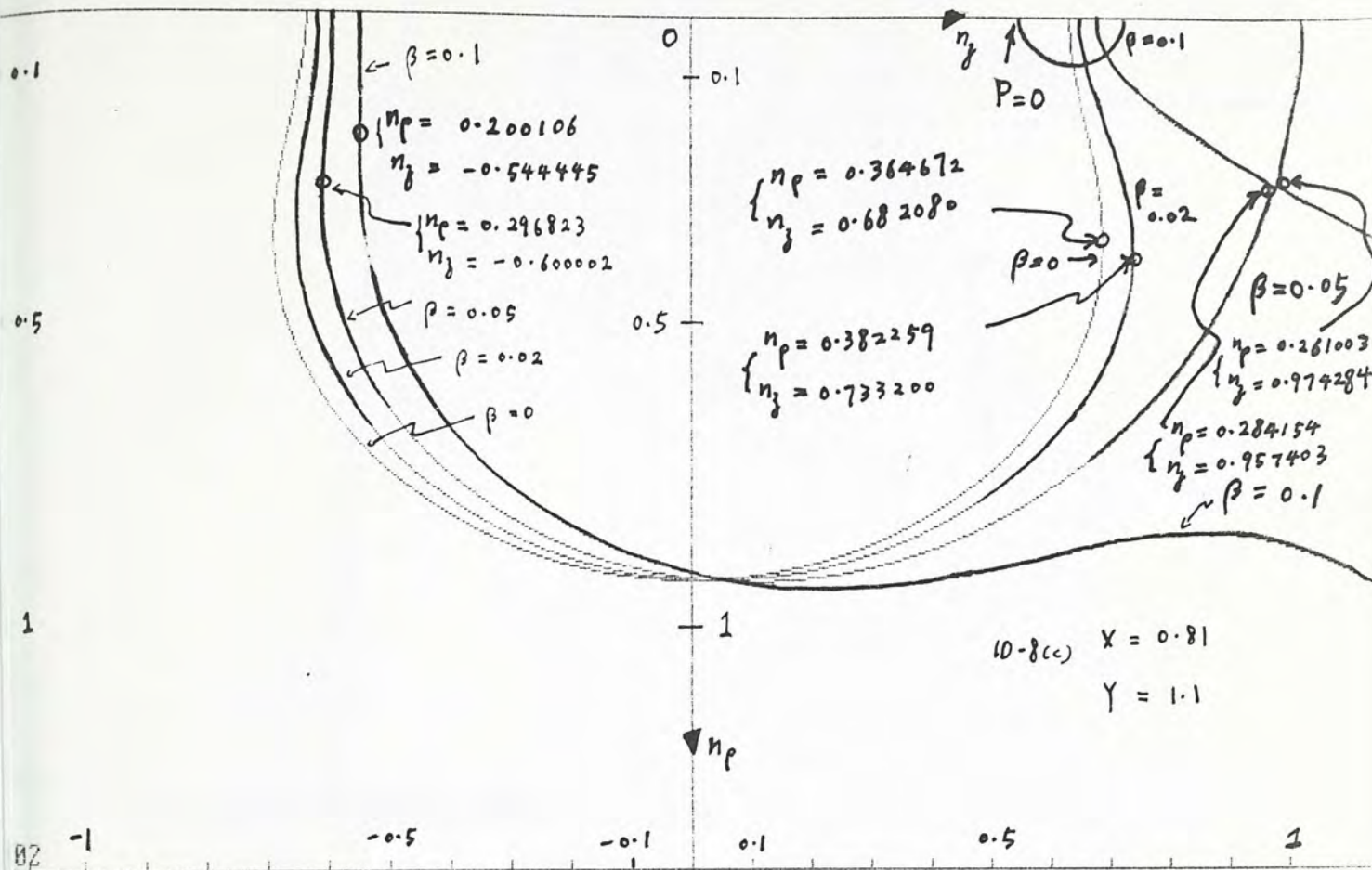
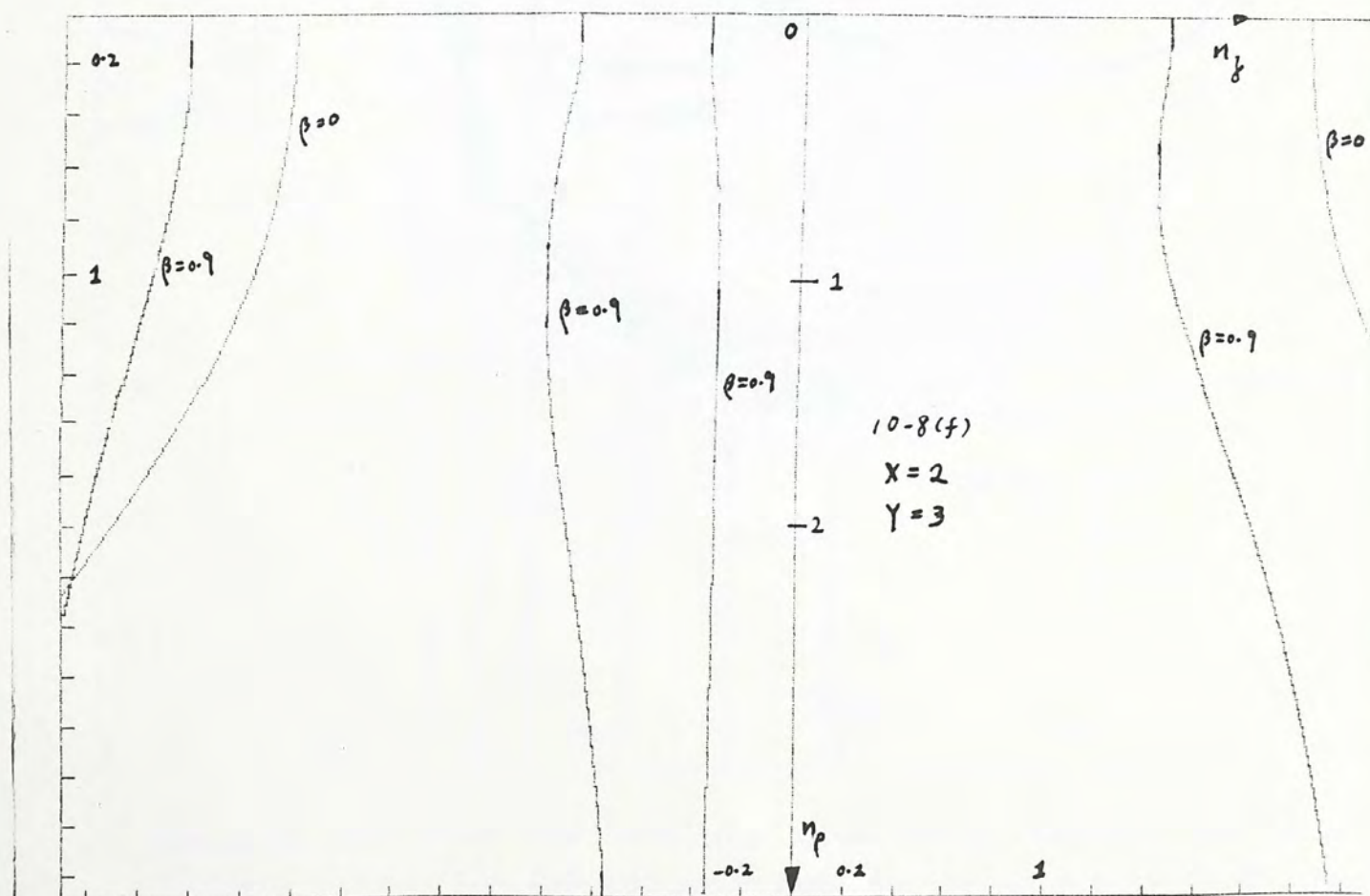
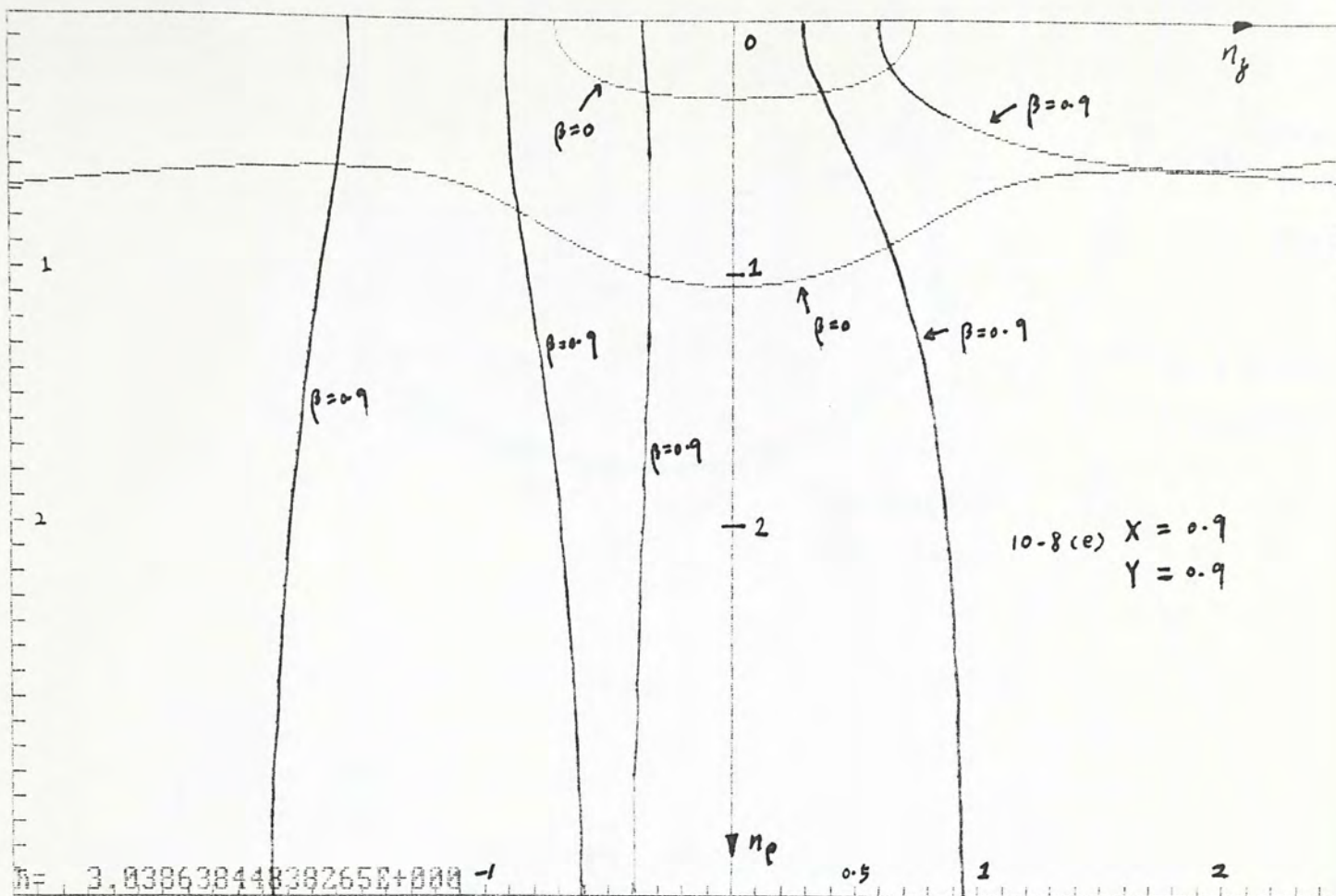


Fig. 10-8 DWS for V along B. Note that now  $n \equiv ck/\omega_0$ . X, Y means  $\Omega_e/\omega_0$  and  $(\omega_{pe}/\omega_0)^2$ .









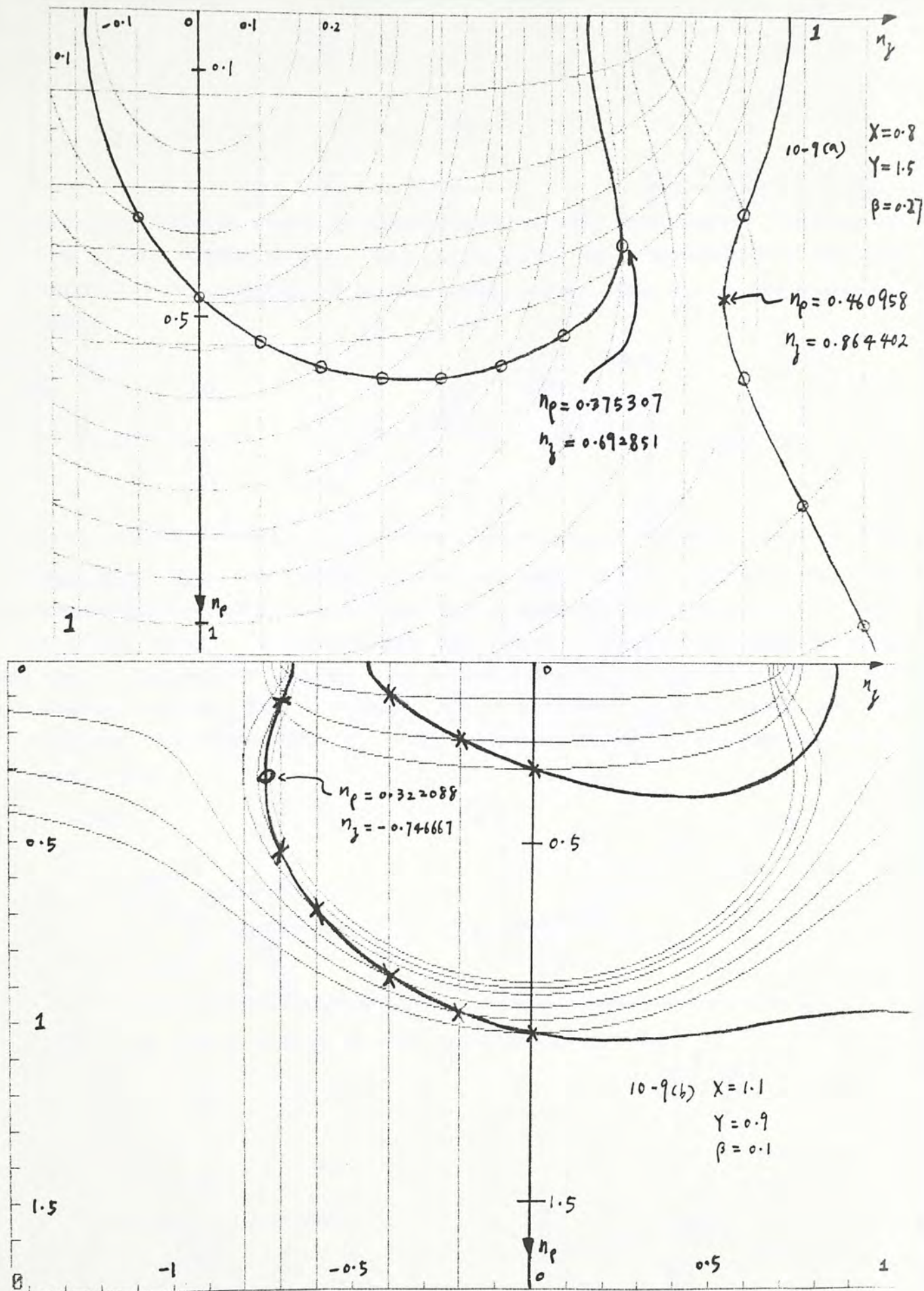


Fig.10-9 DWS with the ordinary wave-vector surface are also plotted so that the DWS can be viewed as the intersection of the Doppler-shifted frequency with the ordinary wave-vector surface.

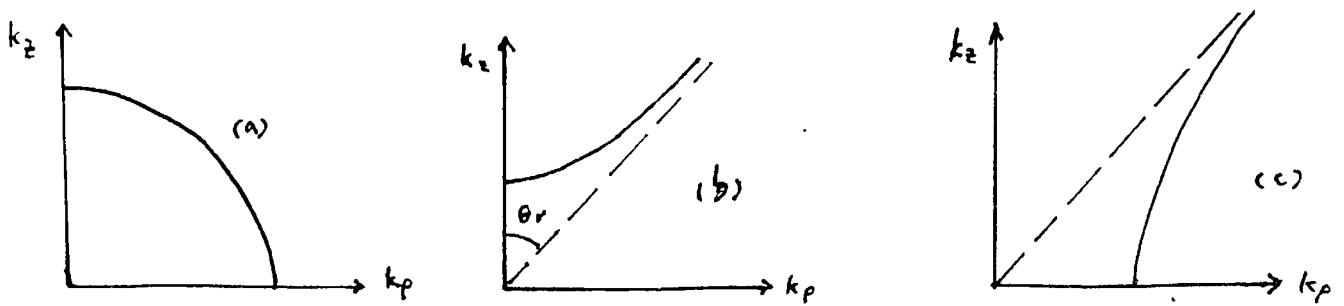


Fig.11-1 Three kinds of topologies of the wave-vector surface of the cold magnetoplasma (a) spheroid, (b) hyperboloid of two sheets, (c) hyperboloid of one sheet. Note that  $\theta_r$  is the resonant angle.

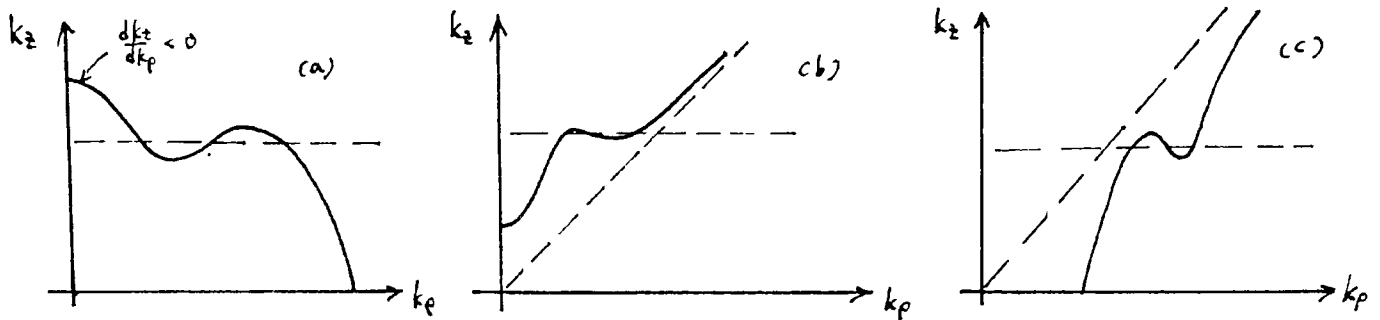


Fig.11-2 For the three kinds of wave-vector surface of cold magnetoplasma : (a) shows that for a spheroid, if  $dk_z/dk_p < 0$  near  $k_p = 0$ , then  $\lambda_\phi = 0$  case can not happen, otherwise there will be more than two  $k_p$  values for one  $k_z$ . (b) shows that for a hyperboloid of two sheets, if  $dk_z/dk_p > 0$  near  $k_p = 0$ , then  $\lambda_\phi = 0$  case can not happen. (c) shows that  $\lambda_\phi = 0$  case never exist in a hyperboloid of one sheet.

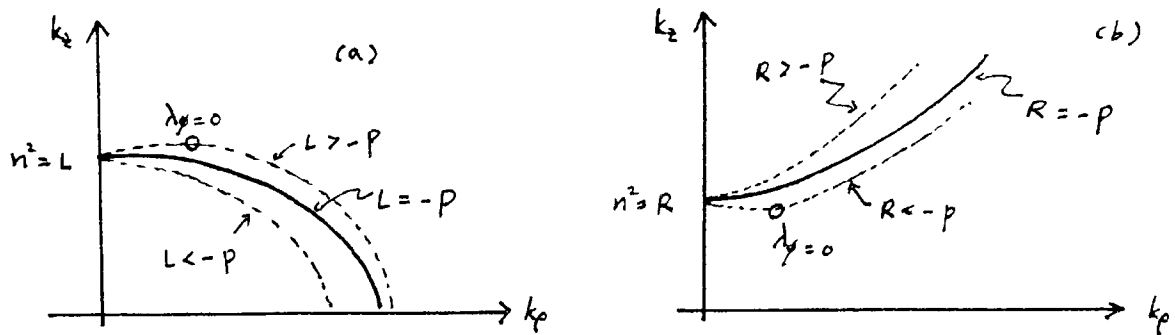


Fig.11-3 For  $L = 0$  or  $R = 0$ ,  $\lambda_\phi = 0$  at  $k_p = 0$ .

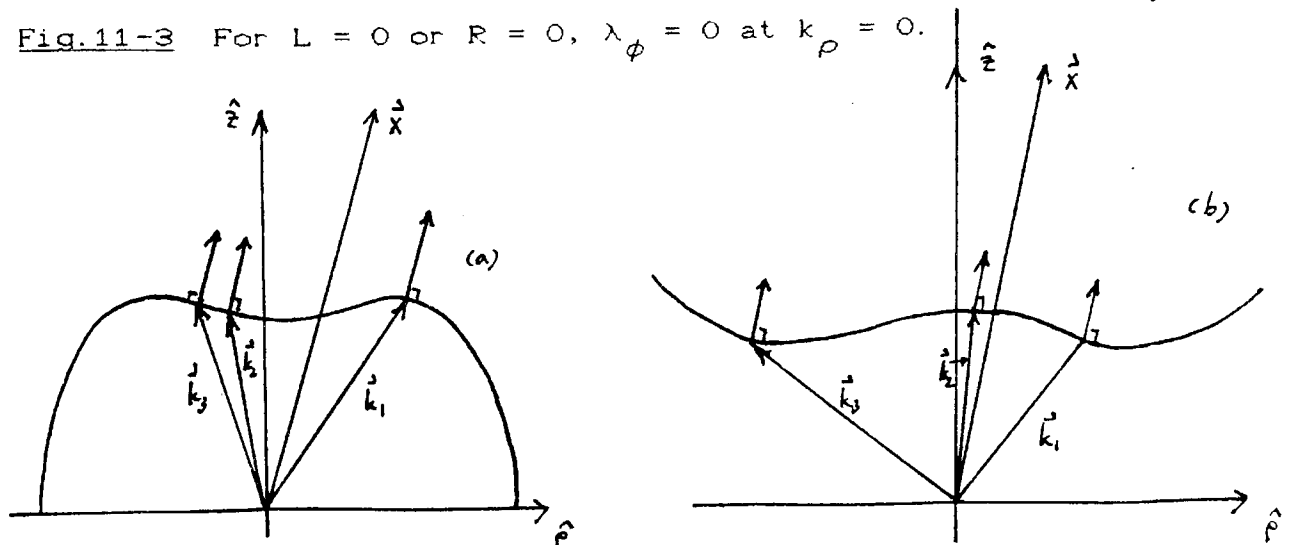


Fig.11-4 Self-mode interference effect may also happen for  $x$  not along the  $\hat{z}$  in the case that there is  $\lambda_\phi = 0$  case on the wave-vector surface.







000488178